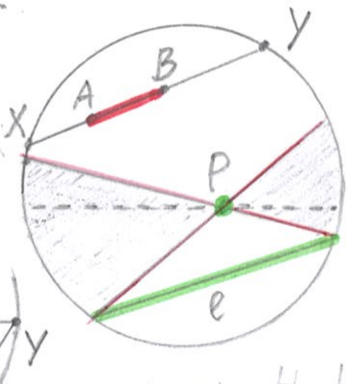
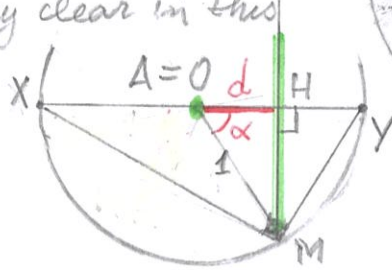
Lecture 9. ISOMORPHIC MODELS OF HYPERBOLIC GEOMETRY

§1. The Cayley-Klein model

Its points are points of the open unit disk $\mathbb{H}^2$, its lines are (open) chords of $\mathbb{H}^2$, the distance between points is given by $\rho(AB) = \frac{1}{2}|\ln\langle ABXY\rangle|$, where $\langle ABXY\rangle$ is the cross ratio of the points A, B, X, Y, wich equals $\frac{AX}{BX} : \frac{AY}{BY}$, and the group defining the geometry is the isometry group $Isom_\rho(\mathbb{H}^2)$.



The parallel postulate is especially clear in this model (see the figure).



The angle of parallelism α depends only on the distance $d = AH$ and is easy to compute if $A = O$:

$$d = \frac{1}{2}\left|\ln\left(\frac{XH}{XA}\cdot\frac{YA}{YH}\right)\right| = \frac{1}{2}\ln\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right) = \ldots \Rightarrow \boxed{\cos\alpha = \operatorname{th} d} \text{ and}$$

$\boxed{e^{-d} = \tan\alpha/2}$ This formula was obtained independently by Bolyai and Lobachevsky

We will prove that this model is isomorphic to the other models of hyperbolic geometry below.

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§2 The half-sphere model

Points are ordinary points of the southern hemisphere S, lines are half-circles obtained as intersections of vertical planes with S. $\longrightarrow$ Fig. 1

The transformation group $\mathcal{M}_S$ will be defined later.

- There is a bijection $\beta_2 : S \to 2\mathbb{H}^2$ that takes lines of S to lines of the Poincaré (doubled) disk model $\longrightarrow$ Fig. 2
- There is a bijection $\beta_+ : S \to \mathbb{C}_+$ that takes lines of S to lines of the Poincaré half plane model $\longrightarrow$ Fig. 3
- There is a bijection $\bar{\beta} : S \to \mathbb{H}^2$ to the Cailey-Klein model ("look from above") that takes lines of S to chords of $\mathbb{H}^2$.

The transformation group $\mathcal{M}_S$ of S can be induced from $\mathcal{M}^+$ using the formula $S \ni x \longmapsto \beta_+^{-1}(g(\beta_+(x)))$, $g \in \mathcal{M}^+$.

We have shown that:

The four models (half-sphere, Cayley-Klein, Poincaré disk, Poincaré half-plane) are isomorphic geometries.

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§3 Hyperbolic functions

Definitions: $sh\,x = (e^x - e^{-x})/2$, $ch\,x = (e^x + e^{-x})/2$, $th(x) = \dfrac{e^x - e^{-x}}{e^x + e^x}$

Formulas: $ch^2 x + sh^2 x = 1$, $sh\,2x = 2 sh\,x\, ch\,x$, $ch\,2x = sh^2 x + ch^2 x$

$sh(x \pm y) = sh\,x\, ch\,y \pm ch\,x\, sh\,y$, $ch(x \pm y) = ch\,x\, ch\,y \pm sh\,x\, sh\,y$

Proofs: substitude definitions in the formulas.

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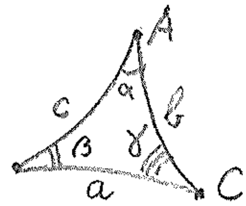
§4. Trigonometry in the hyperbolic plane



$\dfrac{sh\,a}{\sin\alpha} = \dfrac{sh\,b}{sh\,\beta} = \dfrac{sh\,c}{sh\,\gamma}$ (hyperbolic sine theorem)

$ch\,a = ch\,b\, ch\,c - sh\,b\, sh\,c \cos\alpha$ (hyperbolic cosine theorem)

$ch\,a = ch\,b\, ch\,c$ (Pythagoras theorem (a is the hypothenuse))
