On Krull-Schmidt theory[1]

In this note k denotes a field and all rings and algebras are unital and associative.

Recall that a ring A is called *local* if it has an ideal $R \subset A$ such that all elements in $A \setminus R$ are invertible. If this is the case, the quotient ring A/R is a division ring.

Definition 1. An additive category is called *Krull-Schmidt* if

- any object has a decomposition into a finite direct sum of indecomposable objects,
- the endomorphism ring of any indecomposable object is local.

Theorem 2. *Let $\mathcal{A}$ be an additive k-linear idempotent complete category. Suppose that any space $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite-dimensional. Then $\mathcal{A}$ is a Krull-Schmidt category.*

To prove Theorem, we need several lemmas.

Lemma 3. *Let V be a finite-dimensional k-vector space and $A \subset \mathrm{End}(V)$ a subalgebra. Suppose A has no nontrivial idempotents, then any element in A is either invertible (in $\mathrm{End}(V)$) or nilpotent.*

Proof. Take any $a \in A$. Let $\chi_a(t) \in \mathsf{k}[t]$ be the characteristic polynomial of a. We have

$$\chi_a(t) = t^d \cdot f(t)$$

where $f \in \mathsf{k}[t]$, $f(0) \neq 0$. One can find polynomials $p(t), q(t) \in \mathsf{k}[t]$ such that

$$t^d p(t) + f(t)q(t) = 1.$$

Let $a_1 = a^d p(a), a_2 = f(a)q(a) \in A$. Then $a_1 + a_2 = 1$ and $a_1 a_2 = a^d f(a)p(a)q(a) = \chi_a(a)p(a)q(a) = 0$, similarly $a_2 a_1 = 0$. It follows that a_1, a_2 are idempotents. They should be trivial. Suppose $a_1 = a^d p(a) = 1$, it follows that a is invertible (if $d \neq 0$ this is obvious, if $d = 0$ then $\det a = \pm\chi_a(0) \neq 0$ and a is invertible). Suppose $a_2 = f(a)q(a) = 1$, then $f(a)$ is invertible. It follows that $a^d = \chi_a(a)f(a)^{-1} = 0$, and a is nilpotent. □

Lemma 4. *Let A be a finite-dimensional k-algebra. Suppose A has no nontrivial idempotents, then any element in A is either invertible or nilpotent.*

Proof. Consider the left adjoint representation of A: $a \mapsto l_a = a \cdot -$, it is an injective homomorphism

$$l\colon A \to \mathrm{End}_{\mathsf{k}}(A).$$

By Lemma 3 applied to the image of l, for any element $a \in A$ either l_a is nilpotent or invertible. In the first case a is nilpotent. In the second case there exists $b \in A$ such that $ab = l_a(b) = 1$. It follows that $l_a l_b = 1 \in \mathrm{End}_k(A)$, hence $l_b l_a = 1$ and $ba = 1$. Thus a is invertible. □

Lemma 5. *Let A be a ring. Suppose that any element in A is either invertible or nilpotent. Then the set of nilpotents in A is a maximal ideal and A is local.*

Proof. Left to the reader. □

[1]By technical reasons these notes are typed in English. Also, the exposition here is slightly different from one from the lecture

Proof of Theorem 2. The existence of a decomposition is almost clear. We prove it by induction in $\dim(\mathrm{End}(M))$. Clearly, if $M \cong M_1 \oplus M_2$ then $\dim(\mathrm{End}(M)) \geqslant 2$. Hence any M with $\dim(\mathrm{End}(M)) = 1$ is indecomposable. For the induction step, assume $M \cong M_1 \oplus M_2$ and use that

$$\dim(\mathrm{End}(M_1)), \dim(\mathrm{End}(M_2)) < \dim(\mathrm{End}(M)).$$

Now let $M \in \mathcal{A}$ be an indecomposable object. The algebra $A = \mathrm{End}(M)$ is finite-dimensional by assumptions. Also, A has no nontrivial idempotents because M is indecomposable and $\mathcal{A}$ is idempotent complete. Now by Lemmas 4 and 5, the algebra A is local. □

Example 6. Let A be a finite-dimensional k-algebra. Then the categories $\mathrm{mod}{-}A$ and $D^b(\mathrm{mod}{-}A)$ are Krull-Schmidt.

Let X be a projective scheme over k. Then the categories $\mathrm{coh}X$ and $D^b(\mathrm{coh}X)$ are Krull-Schmidt.

Theorem 7. *Let $\mathcal{A}$ be a Krull-Schmidt category. Then decomposition of an object in $\mathcal{A}$ into a direct sum of indecomposable objects is unique in the following sense: if $M = M_1 \oplus \ldots \oplus M_n$ and $M = M'_1 \oplus \ldots \oplus M'_m$ then $n = m$ and up to renumbering of objects one has $M_i \cong M'_i$.*

For the proof we will need a notion of ideal in an additive category.

Definition 8. Let $\mathcal{A}$ be an additive category and $\mathcal{I}$ be a family of morphisms in $\mathcal{A}$. Denote $\mathcal{I}(X,Y) := \mathcal{I} \cap \mathrm{Hom}(X,Y)$ for any $X, Y \in \mathcal{A}$. We say that $\mathcal{I}$ is an *ideal* in $\mathcal{A}$ if for any $f, g \in \mathcal{I}$ and h a morphism in $\mathcal{A}$ we have $-f, f+g, fh, hf \in \mathcal{I}$ as soon as the operations make sense. One-sided ideals are defined similarly.

Lemma 9. *Let $\mathcal{I}$ be an ideal in an additive category $\mathcal{A}$. Let $P_i, Q_i \in \mathrm{Ind}(\mathcal{A})$, $P = \oplus P_i$, $Q = \oplus Q_j$. Let $f \in \mathrm{Hom}(P,Q)$, write it as $f = \sum_{ij} f_{ij}$, where $f_{ij}\colon P_i \to Q_j$. Then $f \in \mathcal{I} \iff f_{ij} \in \mathcal{I}$ for all i, j.*

Proof. It follows clearly from properties of an ideal. □

Definition 10. For an ideal $\mathcal{I}$ in an additive category $\mathcal{A}$ the *quotient category* $\mathcal{A}/\mathcal{I}$ is defined as follows. Its objects are the same as in $\mathcal{A}$ and the Hom spaces are

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(X,Y) := \mathrm{Hom}_{\mathcal{A}}(X,Y)/\mathcal{I}(X,Y).$$

One can see that this is well-defined, $\mathcal{A}/\mathcal{I}$ is an additive category and there is a natural additive functor

$$\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}.$$

Definition 11. Let $\mathcal{A}$ be an additive category. Its *radical* is defined as the ideal generated by all non-invertible morphisms between indecomposable objects. Notation: $\mathcal{R}(\mathcal{A})$.

Proposition 12. *Let $\mathcal{A}$ be a Krull-Schmidt category and $\mathcal{R} = \mathcal{R}(\mathcal{A})$. Let $X, Y \in \mathrm{Ind}(\mathcal{A})$. Then $\mathcal{R}(X,Y)$ consists of all non-invertible morphisms from X to Y.*

Proof. Assume $f \in \mathcal{R}(X,Y)$. If $X \not\cong Y$ there is nothing to prove, if $X \cong Y$ we can assume that $X = Y$. By definition, $f = \sum f_i$ where f_i factors as

$$X \xrightarrow{a_i} X_i \xrightarrow{b_i} Y_i \xrightarrow{c_i} X,$$

where $X_i, Y_i \in \mathrm{Ind}(\mathcal{A})$ and b_i is not invertible. Assume that f_i is invertible, then $b_i a_i$ is a split embedding. Since Y_i is indecomposable, its endomorphism ring has no idempotents, it means that $b_i a_i$ is invertible. Then b_i is a split surjection. Since X_i is indecomposable, it means that b_i is invertible, a contradiction. Hence f_i is not invertible, $f_i \in R(\mathrm{End}(X))$ and thus $f \in R(\mathrm{End}(X))$ is not invertible. □

Let us describe the quotient category of a Krull-Schmidt category by its radical. For an indecomposable object $X \in \mathcal{A}$ denote by T_X the quotient

$$T_X := \mathrm{End}(X)/R(\mathrm{End}(X)),$$

it is a division ring. By Proposition 12 we have

$$\mathrm{End}_{\mathcal{A}/\mathcal{R}(\mathcal{A})}(X) \cong T_X, \quad \mathrm{Hom}_{\mathcal{A}/\mathcal{R}(\mathcal{A})}(X, Y) = 0,$$

for $X \not\cong Y$ indecomposable.

It follows that

Proposition 13. *For a Krull-Schmidt category $\mathcal{A}$ one has*

$$\mathcal{A}/\mathcal{R}(\mathcal{A}) \cong \bigoplus_{X \in \mathrm{Ind}(\mathcal{A})} \mathrm{mod}{-}T_X,$$

where the sum is taken over isomorphism classes of indecomposable objects.

Proof of Theorem 7. Follows from Proposition 13 since $\mathrm{mod}{-}T_X$ is the category of finite-dimensional vector spaces over the division ring and the dimension of such vector spaces is well-defined. □

Remark 14. Let $\mathcal{A}$ be the category of finite-dimensional k-vector spaces V with $\dim V \neq 1$. Then $\mathcal{A}$ is not idempotent complete and a decomposition into indecomposable summands can be not unique. For example, $\mathsf{k}^6 = \mathsf{k}^2 \oplus \mathsf{k}^2 \oplus \mathsf{k}^2 = \mathsf{k}^3 \oplus \mathsf{k}^3$ are two such decompositions. The endomorphism algebras $\mathrm{End}(\mathsf{k}^2) \cong M_2(\mathsf{k})$ and $\mathrm{End}(\mathsf{k}^3) \cong M_3(\mathsf{k})$ of indecomposable objects are not local.

We continue with some properties of Krull-Schmidt categories. For a local ring A we denote its maximal ideal by $R(A)$ and call it the *radical* of A. Denote by $\mathrm{Ind}(\mathcal{A})$ the family of indecomposable objects in $\mathcal{A}$.

Lemma 15. *Let $\mathcal{A}$ be a Krull-Schmidt category, $P, Q_1, \ldots, Q_n \in \mathrm{Ind}(\mathcal{A})$, $P \not\cong Q_i$. Let $P \xrightarrow{f} \oplus Q_i \xrightarrow{g} P$ be morphisms, then $gf \in R(\mathrm{End}(P))$.*

Proof. We have $f = \sum f_i$ where $f_i \colon P \to Q_i$ belongs to $\mathcal{R}(\mathcal{A})$. By Lemma 9 $f \in \mathcal{R}(\mathcal{A})$, hence $gf \in \mathcal{R}(\mathcal{A})$. By Proposition 12 we get that $gf \in R(\mathrm{End}(P))$. □

Lemma 16. *Let $\mathcal{A}$ be a Krull-Schmidt category, $P \in \mathrm{Ind}(\mathcal{A})$, $Q \in \mathcal{A}$. Then a morphism $f \colon P \to Q$ is a split monomorphism iff $f \notin \mathcal{R}(\mathcal{A})$. A similar dual statement holds.*

Proof. If f is a split mono then $gf = 1_P$ for some g. Suppose $f \in \mathcal{R}(\mathcal{A})$, then $1_P \in \mathcal{R}(\mathcal{A})$, a contradiction with Proposition 12. Now assume $f \notin \mathcal{R}(\mathcal{A})$. Write $Q = \oplus Q_i$ with indecomposable Q_i, then $f = \sum f_i$, where $f_i \colon P \to Q_i$. By Lemma 9, $f_i \notin \mathcal{R}(\mathcal{A})$ for some i. By Proposition 12, f_i is an isomorphism. It follows easily that f has a left inverse. □

Lemma 17. *Let $\mathcal{A}$ be a Krull-Schmidt category, $P \in \mathrm{Ind}(\mathcal{A})$, $Q_1, \ldots, Q_n \in \mathcal{A}$, $f_i \colon P \to Q_i$ be some morphisms. Then $f := \oplus f_i \colon P \to \oplus Q_i$ is a split monomorphism iff f_i is a split monomorphism for some i. A similar dual statement holds.*

Proof. By Lemmas 16 and 9, f is a split mono $\iff f \notin \mathcal{R}(\mathcal{A}) \iff f_i \notin \mathcal{R}(\mathcal{A})$ for some $i \iff$ f_i is a split mono for some i. □