On Morita-equivalence[1]

For a ring A, let Mod$-A$ denote the category of right A-modules and mod$-A$ denote the category of finitely generated A-modules. Saying "equivalent" about additive categories we always mean "additive equivalent".

We would like to distinguish the categories of the form Mod$-A$ among general abelian categories.

Let us say that an object G in an abelian category $\mathcal{A}$ is a *generator* if

$$G^{\perp} := \{M \in \mathcal{A} \mid \mathrm{Hom}(G, M) = 0\} = 0.$$

For example, A is a generator in Mod$-A$.

Recall that an object P in an abelian category $\mathcal{A}$ is *projective* if the functor $\mathrm{Hom}(P, -)$ is exact. This is equivalent to the following: for any surjection $X \to Y$ in $\mathcal{A}$ any morphism $P \to Y$ factors via a morphism $P \to X$. Also we recall that an A-module M is projective iff M is a direct summand in a free A-module.

Lemma 1. *Let $\mathcal{A}$ be an abelian category with arbitrary direct sums, let P be a projective generator of $\mathcal{A}$. Then for any object $M \in \mathcal{A}$ there exists a surjection $\oplus P \to M$.*

Proof. Let I be the image of the natural map

$$\oplus_{\mathrm{Hom}(P,M)} P \to M.$$

We claim that $I = M$, assume the contrary. Then (since P is a generator) there exists a nonzero map $\bar{f}\colon P \to M/I$. Since P is projective, $\bar{f}$ lifts to a map $f\colon P \to M$, and the image of f does not lie in I. We get a contradiction with the definition of I. □

Let us say that an object M of an additive category $\mathcal{A}$ is *compact* if for any direct sum $\oplus_{i\in I} M_i$ in $\mathcal{A}$ the natural map

$$\oplus \mathrm{Hom}(M, M_i) \to \mathrm{Hom}(M, \oplus M_i)$$

is an isomorphism. Equivalently, any map $M \to \oplus M_i$ has its image in the finite subsum in $\oplus M_i$. It is easy to see that any finitely generated A-module is a compact object in Mod$-A$.

Proposition 2. *Let $\mathcal{A}$ be an abelian category and A be a ring. Then $\mathcal{A}$ is equivalent to* Mod$-A$ *if and only if the following holds:*

1. *$\mathcal{A}$ admits arbitrary direct sums;*
2. *there exists a compact projective generator P in $\mathcal{A}$ with $\mathrm{End}_{\mathcal{A}} P \cong A$.*

Proof. For "only if", take $P = A$. We now prove "if" part.

Consider the functor

$$\phi\colon \mathcal{A} \to \mathrm{Mod}{-}A, \quad \phi(X) := \mathrm{Hom}(P, X).$$

Left action of $A = \mathrm{End}\, P$ on P makes $\mathrm{Hom}(P, X)$ a right A-module. We will prove that ϕ is an equivalence. First we check ϕ is fully faithful. Note that $\phi(P) = A$. Also note that

$$\mathrm{Hom}_{\mathcal{A}}(P, Y) = \phi(Y) = \mathrm{Hom}_A(A, \phi(Y)) = \mathrm{Hom}_A(\phi(P), \phi(Y))$$

[1] By technical reasons these notes are typed in English. Also, the exposition here is slightly different from one from the lecture

for any $Y \in \mathcal{A}$. Denote

$$\mathcal{A}' := \{X \in \mathcal{A} \mid \phi\colon \operatorname{Hom}_{\mathcal{A}}(X,Y) \xrightarrow{\sim} \operatorname{Hom}_A(\phi(X),\phi(Y)) \quad \text{is an isomorphism for all } Y \in \mathcal{A}\}.$$

Then the subcategory $\mathcal{A}' \subset \mathcal{A}$:

- contains P (by the above);
- is closed under arbitrary direct sums (indeed, P is compact, ϕ preserves direct sums, and $\operatorname{Hom}_{\mathcal{A}}(-,Y)$, $\operatorname{Hom}_A(-,\phi(Y))$ respect direct sums);
- is closed under cokernels (check!).

By Lemma 1, any object X in $\mathcal{A}$ fits into an exact sequence $\oplus P \to \oplus P \to X \to 0$. It follows that $X \in \mathcal{A}'$, $\mathcal{A}' = \mathcal{A}$ and ϕ is fully faithful.

From Lemma 1 is follows that any A-module M fits into an exact sequence

$$\oplus A \xrightarrow{f} \oplus A \to M \to 0.$$

We have $\oplus A \cong \phi(\oplus P)$ and $f = \phi(p)$ for some morphism p in $\mathcal{A}$ (because ϕ is fully faithful). Since ϕ is exact, we have $M \cong \operatorname{coker} f \cong \phi(\operatorname{coker} p)$. Thus ϕ is essentially surjective and an equivalence. □

Problem 1. In the above proof, check that $\mathcal{A}'$ is closed under cokernels.

There is a similar statement for the categories of finitely generated modules.

Proposition 3. *Let $\mathcal{A}$ be an abelian category and A be a ring. Then $\mathcal{A}$ is equivalent to* $\mathrm{mod}{-}A$ *if and only if there exists a projective generator P in $\mathcal{A}$ with $\operatorname{End}_{\mathcal{A}} P \cong A$ such that any object in $\mathcal{A}$ is a quotient of P^N for some $N \in \mathbb{N}$.*

Problem 2. Prove Proposition 3

Problem 3. Let $\phi\colon \mathrm{Mod}{-}A \to \mathrm{Mod}{-}B$ be an additive equivalence, then ϕ restricts to an equivalence between $\mathrm{mod}{-}A$ and $\mathrm{mod}{-}B$.

Definition 4. Rings A and B are called *Morita-equivalent* if the categories $\mathrm{Mod}{-}A$ and $\mathrm{Mod}{-}B$ are (additive) equivalent.

Remark 5. Rings A and B are Morita-equivalent if and only if the categories $\mathrm{mod}{-}A$ and $\mathrm{mod}{-}B$ are equivalent. "Only if" part follows from Problem 3, for the "if" part assume $\mathrm{mod}{-}A \cong \mathrm{mod}{-}B$. Use Proposition 3 to get a projective generator P in $\mathrm{mod}{-}B$ with $\operatorname{End}_B P \cong A$. Ensure that P is compact in $\mathrm{Mod}{-}B$ and generates it, then use Proposition 2.

Remark 6. From the above it follows that the rings A and B are *Morita-equivalent* if and only if there exists a projective generator P in $\mathrm{mod}{-}B$ with $\operatorname{End}_B P \cong A$.

Problem 4. Prove that any equivalence $\phi\colon \mathrm{Mod}{-}B \to \mathrm{Mod}{-}A$ has the form $\operatorname{Hom}_B(P,-)$ for some A-B-bimodule P.

Now we turn to algebras. Let k be a field. By an algebra we mean an associative unital k-algebra.

Let A be a finite-dimensional algebra. Then A is noetherian, the category $\mathrm{mod}{-}A$ of finitely generated (or of finite dimensional over k) A-modules is abelian and Krull-Schmidt.

Recall that an object X of an abelian category $\mathcal{A}$ is *simple* if it has no subobjects except for 0 and X. An object is *semisimple* if it is a direct sum of simple objects. An object *has finite length* if it has a finite filtration with simple factors. By Jordan-Hoelder theorem, for any two such filtrations the length and the collection of factors is the same. Hence, length of an object is well-defined. An abelian category is said to have *finite length* if all its objects have finite length.

Clearly, the category mod$-A$ for a finite-dimensional algebra has finite length.

Note also that for an indecomposable object N in mod$-A$ the radical of the ring $\operatorname{End} N$ is nilpotent (it follows from the first lecture).

For $N \in \operatorname{Ind}(\text{mod}-A)$ let us denote $T(N) := (\operatorname{End} N)/R(\operatorname{End} N)$, it is a division ring.

Take any $N \in \text{mod}-A$, decompose N into indecomposables:

$$N \cong \bigoplus_{i=1}^{n} N_i^{d_i},$$

where N_i-s are pairwise non-isomorphic indecomposable. Then

$$\operatorname{End}_{\text{mod}-A} N \cong \operatorname{End}(\oplus N_i^{d_i}) \cong \oplus_{i,j} \operatorname{Hom}(N_i^{d_i}, N_j^{d_j}) \cong \oplus_{i,j} M_{d_j \times d_i}(\operatorname{Hom}(N_i, N_j)).$$

It is convenient to consider the full subcategory $\mathcal{A}_0 \subset \text{mod}-A$ with objects $N_1^{d_1}, \ldots, N_n^{d_n}$ and to identify A with $\mathcal{A}_0$ or just with the full subcategory $\{\oplus_{i=1}^n N_i^{d_i}\} \subset \text{mod}-A$.

Definition 7. The *radical* $R(A)$ of a finite-dimensional algebra A is defined as the biggest nilpotent ideal in A (one can show that such an ideal exists). Algebra A is called *semisimple* if $R(A) = 0$.

Proposition 8. *In the above convention,*

$$R(\operatorname{End} N) = \mathcal{R}(\text{mod}-A) \cap \mathcal{A}_0 = \oplus_{i \neq j} M_{d_j \times d_i}(\operatorname{Hom}(N_i, N_j)) \bigoplus \oplus_i M_{d_i}(R(\operatorname{End} N_i)).$$

Proof. The second equality follows from the first lecture. To prove the first one, denote the middle term by I. Denote $T_i := T(N_i)$.

The quotient functor

$$\text{mod}-A \to (\text{mod}-A)/\mathcal{R}(\text{mod}-A)$$

induces a surjective homomorphism

$$\operatorname{End}_{\text{mod}-A} N \to \operatorname{End}_{(\text{mod}-A)/\mathcal{R}(\text{mod}-A)} N \cong \prod_i M_{d_i}(T_i).$$

Its kernel is exactly I. Since the ring $\prod_i M_{d_i}(T_i)$ has no nonzero nilpotent ideals, it suffices to check that I is nilpotent.

To do this, we prove that any composition of certain number of morphisms in $\mathcal{R}(\text{mod}-A) \cap \mathcal{A}_0$ vanishes. Choose m such that $R(\operatorname{End} N_i)^m = 0$ for any i. Now take a composition of any mn morphisms in $R(\text{mod}-A)$ between objects of $\mathcal{A}_0$. By Dirichlet principle, they can be grouped to contain a product of m cycles starting and ending at some fixed object $N_i^{d^i} \in \mathcal{A}_0$. Any such cycle (as a morphism) is in $M_{d_i}(R(\operatorname{End} N_i))$. By our assumption, the product of cycles is zero. □

Corollary 9. *In the above notation, assume* $\operatorname{End} N$ *is semisimple. Then* $\operatorname{Hom}(N_i, N_j) = 0$ *for* $i \neq j$ *and each* $\operatorname{End} N_i$ *is a division ring.*

Corollary 10. *In the above notation*

$$(\operatorname{End} N)/R(\operatorname{End} N) \cong \prod_{i=1}^{n} M_{d_i}(T_i).$$

Let us apply the above construction to the right A-module A. Note that $\operatorname{End}_{\mathrm{mod}-A} A \cong A$, where A acts on the right A-module A by left multiplications.

Proposition 11. *Let $A = \oplus_{i=1}^{n} P_i^{d_i}$ be the decomposition into indecomposables, then*

$$A \cong \operatorname{End}(\oplus P_i^{d_i}) \cong \oplus_{i,j} M_{d_j \times d_i}(\operatorname{Hom}(P_i, P_j)),$$
$$R(A) = \oplus_{i \neq j} M_{d_j \times d_i}(\operatorname{Hom}(P_i, P_j)) \bigoplus \oplus_i M_{d_i}(R(\operatorname{End} P_i)),$$
$$A/R(A) \cong \prod_{i=1}^{n} M_{d_i}(T(P_i)).$$

Now let us give some other definitions of a semisimple algebra.

Proposition 12. *The following conditions on a finite-dimensional algebra A are equivalent.*

1. *A is a semisimple algebra;*
2. *for any finitely generated A-modules $M' \subset M$ the submodule M' is a direct summand;*
3. *any finitely generated A-module is semisimple;*
4. *A is semisimple as a right A-module;*
5. *$A \cong \prod_{i=1}^{n} M_{d_i}(T_i)$ for some division algebras T_i.*

Proof. Equivalences between (2),(3) and (4) are by general properties of semisimple objects.

For (4) $\Rightarrow$ (5), use Proposition 11. Note that P_i-s are simple modules and use Schur lemma: $\operatorname{Hom}(P_i, P_j) = 0$ for $i \neq j$ and $\operatorname{End} P_i$ is a division algebra.

For (5) $\Rightarrow$ (1) it suffices to check that the radical of a matrix algebra $M_d(T)$ over a division ring T has no nontrivial ideals. This is easy.

For (1) $\Rightarrow$ (4), use Corollary 9 and Proposition 11. To check that P_i-s are simple, assume $M \subset P_i$ is a submodule. Choose a projective covering $\oplus P_j^{m_j} \to M$ and consider the composite map $f\colon \oplus P_j^{m_j} \to M \to P_i$. By Corollary 9, f is either zero or surjective. Hence P_i has no nontrivial submodules. □

Now we are going to find all algebras which are Morita-equivalent to a given finite-dimensional algebra A. Write

$$A = \oplus_{i=1}^{n} P_i^{d_i}$$

where P_i are pairwise non-isomorphic indecomposable projective. Note that any indecomposable projective A-module is (by Krull-Schmidt theorem, as a summand in a free module) one of $P_1, \ldots, P_n$. A sum $\oplus_{i=1}^{n} P_i^{m_i}$ is a generator of Mod$-A$ iff all $m_i > 0$. Indeed, otherwise (by Lemma 1) there exists a surjection $\oplus_{j \neq i} P_j^{n_j} \to P_i$. Since P_i is projective, this surjection splits, P_i is a direct summand in $\oplus_{j \neq i} P_j^{n_j}$, what contradicts to Krull-Schmidt theorem. Using Remark 6 we get

Proposition 13. *Let A be a finite-dimensional algebra. Write*

$$A = \oplus_{i=1}^{n} P_i^{d_i}$$

where $P_1, \ldots, P_n$ are pairwise non-isomorphic indecomposable projective modules. Then an algebra is Morita-equivalent to A if and only if it is isomorphic to the algebra

$$\mathrm{End}(\oplus_{i=1}^{n} P_i^{m_i})$$

for some $m_1, \ldots, m_n > 0$.

In particular, an algebra which is Morita-equivalent to a finite-dimensional one, is also finite-dimensional.

Among algebras from Proposition 13 there is "the smallest one" with all $m_i = 1$. It is called basic.

Definition 14. A finite dimensional algebra A is called *basic* if $A \cong \oplus_{i=1}^{n} P_i$ in mod$-A$ with $P_1, \ldots, P_n$ indecomposable non-isomorphic.

Proposition 15. *A finite-dimensional algebra A is basic if and only if $A/R(A)$ is a direct product of division algebras.*

Proof. Write $A = \oplus P_i^{d_i}$, then $A/R(A) \cong \prod M_{d_i}(T(P_i))$ by Proposition 11. If A is basic then all $d_i = 1$ and $A/R(A) \cong \prod T(P_i)$ is a product of division algebras. If A is not basic and some $d_i > 1$ then $\prod M_{d_i}(T(P_i))$ has nilpotents and is not a product of division algebras. □

Proposition 16 (Gabriel)**.** *Any finite-dimensional algebra A is Morita-equivalent to a unique basic algebra.*

Proof. By Proposition 13, all algebras Morita-equivalent to A are of the form $B = \mathrm{End}(\oplus_{i=1}^{n} P_i^{m_i})$, where P_i-s are indecomposable projective A-modules. By Corollary 10, $B/R(B) \cong \prod M_{m_i}(T(P_i))$, this a product of division algebras iff all $m_i = 1$. Hence B is basic only for all $m_i = 1$. □