** *****

INTRODUCTION TO TOPOLOGY

Lecture Notes in Simple English (IUM, Fall 1999)

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Lecture 1 WHAT IS TOPOLOGY?

What is topology? There are many answers to this question. Examples:

(1) topology is the mathematics of continuity;
(2) topology is the study of the deep properties of geometric figures;
(3) topology is the theory of topological spaces;
(4) topology is rubber sheet geometry.

Let us look at these answers in more detail.

§1. Continuity

Let $\mathbb{R}$ be the set of real numbers. A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous at the point* $x_0 \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x_0) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$ such that $|x_0 - x| < \delta$. The function f is called *continuous* if it is continuous at all points $x \in \mathbb{R}$. (This is elementary calculus.)

Let $\mathbb{R}^n$ be n-dimensional space. By $O_r(p)$ denote the *open ball* of radius $r > 0$ and center $p \in \mathbb{R}^n$, i.e., the set $O_r(p) := \{q \in \mathbb{R}^n | d(p,q) < r\}$, where d is the distance in $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *continuous at the point* $p_0 \in \mathbb{R}^n$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(p) \in O_\varepsilon(f(p_0))$ for all $p \in O_\delta(p_0)$. The function f is called *continuous* if it is continuous at all points $p \in \mathbb{R}^n$. (This is advanced calculus.)

A set $G \subset \mathbb{R}^n$ is called *open in* X if for any point $g \in G$ there exists a $\delta > 0$ such that $O_\delta(g) \subset U$. Let $X \subset \mathbb{R}^n$. A subset $U \subset X$ is called *open in* X if for any point $u \in U$ there exists a $\delta > 0$ such that $O_\delta(u) \cap X \subset U$. Let X and Y be subsets of $\mathbb{R}^n$. A map $f : X \to Y$ is called *continuous* if the preimage of any open set is an open set, i.e.,

$$V \text{ is open in } Y \implies f^{-1}(V) \text{ is open in } X.$$

(This is basic topology.)

Let us compare the three definitions of continuity. Clearly, the topological definition is the simplest. Also, the topological definition gives the simplest proofs. Here is an example.

Theorem 1. *The composition of continuous maps is a continuous map. In more detail, if X, Y, Z are subsets of $\mathbb{R}^n$, $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then their composition, i.e., the map $h = g \circ f : X \to Z$ given by $h(x) := g(f(x))$, is continuous.*

Proof. Let $W \subset Z$ be open. Then the set $V := f^{-1}(W) \subset Y$ is open (because f is continuous). Therefore, the set $U := g^{-1}(V) \subset X$ is open (because g is continuous). But $U = h^{-1}(W)$. $\square$

Compare this proof with the proof of the corresponding theorem in elementary calculus. This proof is much simpler.

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The notion of open set is fundamental in topology. Other basic notions (continuity, closed set, closure, interior, etc.) are defined by using open sets.

Let $X \subset \mathbb{R}^n$. A subset $F \subset X$ is called *closed* if $X - F$ is open in X. Let $A \subset X$; an *interior point* of A is a point $x \in A$ such that there exists an open set U in X such that $x \in U$ and $U \subset A$. The set of all interior points of A is called the *interior* of A in X and is denoted by $\operatorname{Int} A$. An *isolated point* of A in X is a point $a \in A$ such that there exists an open set U in X such that $U \cap A = a$. A *boundary point* of A in X is a point $x \in X$ such that any open set $U \ni x$ in X contains points of A and points not in A, i.e., $U \cap A \neq \varnothing$ and $U \cap (X - A) \neq \varnothing$; the boundary of A is denoted by $\operatorname{Bd}(A)$ or ∂A. The union of A and all the boundary points of A is called the *closure* of A in X and is denoted by $\operatorname{Clos}(A, X)$ (or $\operatorname{Clos}(A)$, or $\overline{A}$, if X is clear from the context).

Theorem 2. (a) *A set A is closed iff A contains all the boundary points of A.*

(b) *The interior of a set A is the largest (by inclusion) open set contained in A.*

(c) *The closure of a set A is the smallest (by inclusion) closed set containing A.*

(d) *The boundary of a set A is the difference between the closure of A and the interior of A:* $\operatorname{Bd}(A) = \operatorname{Clos}(A) - \operatorname{Int}(A)$.

The proof is straightforward. □

§2. Topological equivalence

What are the "deep properties" (see (2) in §1) of "geometric figures"? In this section we answer this question.

Let X and Y be "geometric figures", i.e., arbitrary subsets of $\mathbb{R}^n$. Then X and Y are called *topologically equivalent* or *homeomorphic* if there exists a *homeomorphism* of X onto Y, i.e., a continuous bijective map $h : X \to Y$ such that the inverse map h^{-1} is continuous.

For the topologist, homeomorphic figures are the same figure, a circle is the same as a square, a coffee cup is the same as a doughnut. If a property does not change under any homeomorphism, then this property is called *topological*. The "deep properties" of (2) in §1 are the topological properties. Two important examples of topological properties are compactness and arcwise connectedness.

2.1. Compactness. A family $\{U_\alpha\}$ of open sets in $X \subset \mathbb{R}^n$ is called an *open covering* of X if this family covers X, i.e., if $\cup_\alpha U_\alpha \supset X$. A *subcovering* of $\{U_\alpha\}$ is a subfamily $\{U_{\alpha_\beta}\}$ such that $\cup_\beta U_{\alpha_\beta} \supset X$, i.e., the subfamily also covers X. The set X is called *compact* if every open covering of X contains a finite subcovering.

Theorem 3. *The continuous image of a compact set is compact. In more detail, if the map $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

The proof is staightforward. □

Corollary 1. *Compactness is a topological property.* □

Theorem 4. *A set $X \subset \mathbb{R}^n$ set is compact if and only if X is closed and bounded.*

The proof is usually given in elementary calculus courses.

2.2. Arcwise connectedness. A set $X \subset \mathbb{R}^n$ is called *arcwise connected* if any two points of X can be joined by an arc, i.e., if for any $x, y \in X$ there exists a continuous map $\varphi : [0, 1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

Theorem 5. *The continuous image of an arcwise connected set is arcwise connected. In more detail, if the map $f : X \to Y$ is continuous and X is arcwise connected, then $f(X)$ is arcwise connected.*

Proof. Let $y_1, y_2 \in f(X)$. Let $X_1 := f^{-1}(y_1)$ and $X_2 := f^{-1}(y_2)$. Let x_1 and x_2 be arbitrary points of X_1 and X_2, respectively. Then there exists a continuous map $\varphi : [0, 1] \to X$ such that $\varphi(0) = x_1$ and $\varphi(1) = x_2$ (because X is arcwise connected). Let $\psi : [0, 1] \to f(X)$ be defined by $\psi := f \circ \varphi$. Then ψ is continuous (by Theorem 1) and $\psi(0) = y_1$ and $\psi(1) = y_2$. □

Corollary 2. *Arcwise connectedness is a topological property.* □

§3*. (Abstract) topological spaces

We will not study abstract topological spaces in this course. So the material of this section is optional.

By definition, an *(abstract) topological space* $(X, \mathcal{T} = \{U_\alpha\})$ is a set X of arbitrary elements $x \in X$ (called *points*) and a family $\mathcal{T} = \{U_\alpha\})$ (called the *topology of the space* X) of subsets of X (called *open sets*) such that

(1) X and $\varnothing$ are open;
(2) if U and V are open, then $U \cap V$ is open;
(3) if $\{V_\beta\}$ is a family of open sets, then the set $\cup_\beta V_\beta$ is open.

Any set $X \subset \mathbb{R}^n$ is a topological space if the family of open sets is defined as in §1. (The proof is a straightforward exercise.) All the definitions from §§1,2 are good for any topological space (not only for subsets of $\mathbb{R}^n$), because they only use the notion of open set. All the theorems (and their proofs) from §§1,2 are also good.

Here is an example of an abstract topological definition and theorem. A topological space X is called *connected* if it is not the union of two open, closed, nonempty, and nonintersecting sets, i.e., $X = A \cup B$, where A and B are both open, closed, and nonempty, implies $A \cap B \neq \varnothing$.

Theorem 6. *The continuous image of a connected topological space is connected.*

The proof is short. But it uses an abstract, unfamiliar argument. Can you find it?

§4. "Rubber sheet geometry"

A nice example of a topological space is a surface in $\mathbb{R}^3$, e.g. the sphere, the torus, the cylinder, or the Moebius band. In topology, you can think that the surface is made out of thin rubber material. The surface can be moved, bent, stretched, contracted (but not torn): for the topologist it will still be the same surface. We will study topological surfaces in subsequent lectures. Here we only note that, intuitively, a topological equivalence is simply a transformation of the surface that can be done physically with thin rubber models of the surface.

Lecture 2 TOPOLOGICAL THEOREMS IN THE PLANE

In this lecture we will formulate (and sometimes prove) some important topological theorems in two-dimensional topology, i.e., in the topology of the plane $\mathbb{R}^2$.

§1. A Fixed Point Theorem

Fixed point theorems are very important for applications of topology. Such theorems are used to prove the existence of solutions of various types of equations, e.g. differential equations. In this lecture we shall prove a simple fixed point theorem and see how it can be applied to solve systems of transcendental equations.

Let $f : X \to X$ be a map; if $f(x_0) = x_0$, then the point x_0 is called a *fixed point* of the map f. We say that the set X has the *fixed point property* if for any continuous map $f : X \to X$ there exists a fixed point.

Remark 1. The fixed point property is a topological property. (The proof of this fact is straightforward.)

Remark 2. Fixed points are closely related to solutions of equations. Indeed, if $f(x) = 0$ is an equation, then x_0 *is a solution of the equation* $f(x) = 0$ *if and only if* x_0 *is a fixed point of the map* F *given by the formula* $F(x) := f(x) + x$. Proof:

$$F(x_0) = x_0 \iff f(x_0) + x_0 = x_0 \iff f(x_0) = 0. \quad \square$$

Let $x \in \mathbb{R}^n$; then $|x|$ denotes the distance between the point x and the origin O of coordinates in $\mathbb{R}^n$ (or, if we regard x as a vector, then $|x|$ is the *length* of the vector x). Now let us define three standard topological objects:

- the *n-disk* $D^n := \{x \in \mathbb{R}^n : |x| \leqslant 1\}$;
- the *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}$;
- the *n-ball* $B^n := \{x \in \mathbb{R}^n : |x| < 1\}$.

Theorem 1 [Brouwer]. *The disk D^2 has the fixed point property: for any continuous map $f : D^2 \to D^2$ there exists a fixed point $x_0 \in D^2$, $f(x_0) = x_0$.*

Remark 3. The ball and the sphere do not have the fixed point property. (The proof is an exercise.)

To prove Theorem 1, we need a definition and another theorem. Let X be a space and $Y \subset X$; a continuous map $r : X \to Y$ is called a *retraction* of X onto Y if $r(y) = y$ for all $y \in Y$.

Theorem 2. *There is no retraction of the disk D^2 onto its boundary S^1.*

Proof of Theorem 1 using Theorem 2. Assume the contrary: suppose there exists a continuous map $f : D^2 \to D^2$ without any fixed points. Let us construct a map $r : D^2 \to S^1$. Suppose $x \in D^2$; then $f(x) \neq x$ (because there are no fixed points). Consider the ray $[f(x), x)$ and denote $r(x) := S^1 \cap [f(x), x)$. It is easy to prove that r is continuous (just do it!). Further, $r(y) = y$ for any $y \in S^1$ (why?). So r is a retraction of D^2 onto S^1. But this is impossible by Theorem 2. $\square$

Proof* of Theorem 2. This proof is rather delicate. It is considered optional in this course.

Assume the contrary: let $r : D^2 \to D^2$ be a retraction of D^2 onto its boundary S^1. Let $s \in S^1$ be the "South Pole" of S^1 and $X := r^{-1}(s)$. Then X is bounded and closed (why?), and so X is compact (by Theorem 6 from Lecture 1). Let $U \subset S^1$ be an small open set containing s, say $U := S^1 \cap O_{0.001}(s)$. The map r is continuous; therefore for every point $x \in X$ there exists an equilateral triangle Δ_x with center x such that $r(\Delta_x) \subset U$. Then the family of open triangles $\{\operatorname{Int}\Delta_x | \, x \in X\}$ is a covering of X. But X is compact; therefore there exists a finite subcovering

$$\operatorname{Int}\Delta_{x_1} \cup \operatorname{Int}\Delta_{x_2} \cup \cdots \cup \operatorname{Int}\Delta_{x_k} =: W \supset X.$$

Let a_1 be a boundary point of $W \cap S^1$ in S^1. The point a_1 lies on a side of one of the triangles Δ_{x_i}. Let a_2 be the first intersection point of the side of Δ_{x_i} with another triangle from the subcovering; let a_3 be the first intersection point of the side of the second triangle with another triangle from the subcovering; and so on. We obtain a sequence of segments $[a_1, a_2], [a_2, a_3], \ldots$. The number of sides of the triangles is finite; therefore our sequence is finite,

$$[a_1, a_2],\ [a_2, a_3], \ldots,\ [a_{p-1}, a_p],$$

and so it must end on U, i.e., $a_p \in U$.

Let us call the set $B := [a_1, a_2] \cup \cdots \cup [a_{p-1}, a_p]$ a *limiting broken line*. Obviously, $B \cap X = \varnothing$.

If b_1 is another boundary point of $W \cup S^1$ in S^1, then we can construct another limiting broken line. Obviously, there is a finite number of limiting broken lines.

Lemma 1. *At least one limiting broken line has endpoints on different sides of s.*

The lemma follows from the Jordan Curve Theorem (see §2 below).

Lemma 2. *If B is a limiting broken line with endpoints on different sides of s, then there exists a point $b \in B$ such that $r(b) = s$.*

This lemma easily follows from the "Intermediate Value Theorem" (proved in calculus courses).

But now the result of Lemma 2 contradicts the fact that $B \cap X = \varnothing$, because $X := r^{-1}(s)$. The theorem is proved. $\square$

Let us describe an application of the Fixed Point Theorem (Theorem 1). Consider a system of equations of the form:

$$f(x, y) = 0, \quad g(x, y) = 0, \tag{1}$$

where f and g are continuous real-valued functions. Consider the map F from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by

$$(x, y) \mapsto (f(x, y), g(x, y)) + (x, y).$$

Then (x_0, y_0) is a solution of system (1) if and only if (x_0, y_0) is a fixed point of the map F. Therefore Theorem 1 implies

Corollary 1. *If F maps a disk $D^2 \subset \mathbb{R}^2$ into itself, then system* (1) *has a solution.*

Example. By Corollary 1, the system

$$\begin{cases} -x\sin^2\left(\dfrac{xy^2}{17}+0.1\right)=0, \\ \dfrac{1}{3}\sin^2 x^2\sqrt{|\cos^3|x||}-y=0 \end{cases}$$

has a solution. The proof is an exercise.

Theorem 1 is a two-dimensional fixed point theorem. A similar n-dimensional theorem is true for D^n (for any natural number n.) The proof is much more difficult and uses a branch of topology called "homology theory". But the really important fixed point theorems are "infinite-dimensional" and, of course, even more difficult.

§2. The Jordan Curve Theorem

A closed curve in the plane is said to be simple if it has no self-intersections. More precisely, a subset C of the plane, $C \subset \mathbb{R}^2$, is called a *simple closed curve* if there exists a homeomorphism γ of the circle S^1 onto C.

Theorem 3 [Jordan Curve Theorem]. *Any closed curve $C \subset \mathbb{R}^2$ divides the plane into exactly two regions, the inner region and the outer region. This means that $\mathbb{R}^2 \backslash C = B_1 \cup B_2$, $B_1 \cap B_2 = \varnothing$, where any two points of the same region (B_1 or B_2) can be joined by a broken line inside that region, and any broken line joining two points from different regions intersects the curve C.*

The proof of this theorem is difficult and will not be given in this course. But we shall often use this theorem (we already used it in the proof of Theorem 2 above). We shall also use (also without proof) a stronger version of this theorem:

Theorem 4 [Schoenflies Theorem]. *Let $\gamma : S^1 \to C = \gamma(S^1) \subset \mathbb{R}^2$ be a simple closed curve. Then there exists a homeomorphism $h_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that the boundary $\operatorname{Fr} D^2$ of the unit disk $D^2 := \{x \in \mathbb{R}^2\}$ is mapped to C, i.e., $\gamma(\operatorname{Fr} D^2) = C$. Moreover, h_1 is part of a continuous family of homeomorphisms $\{h_t | t \in [0,1]\}$, where h_0 is the identity map.*

§3. Planar Graphs and the Euler Characteristic

A *graph* is a finite set V of points, called *vertices* or *nodes*, and a finite E set of arcs, called *edges*, joining the vertices, i.e., arcs whose endpoints are vertices. If the two endpoints of an edge coincide, this edge is called a *loop*; if more than one edge joins two vertices, we say that we have a *multiple edge* joining these vertices. The *degree* of a vertex of a graph is the number of edges with endpoints at this vertex (a loop at the vertex adds 2 to the degree). Two graphs are called *isomorphic* if there is a bijection between the vertices and a bijection between the edges such that the endpoints of any edge correspond to the endpoints of the corresponding edge. A graph G' is called a *subgraph* of the graph G if it is a part of G, more precisely if each vertex of G' belongs to G and each edge of G' is an edge of G joining vertices belonging to G'.

A *path* in a graph G joining the vertices v and v' is a sequence of vertices and edges $v = v_1, e_1, v_2, \ldots, e_k, v_{k+1} = v'$ such that for each i, v_i and v_{i+1} are the endpoints of e_i. A *circuit* in a graph G is a path joining a vertex to itself; a circuit is called *simple* if it has no self-intersections, i.e., all the vertices of the circuit are distinct. If any two points of a graph can be joined by a path, then this graph is called *connected*. A connected graph without loops and without circuits is called a *tree*.

A graph is called *planar* if it can be drawn in the plane $\mathbb{R}^2$ so that the edges intersect only at their common vertices. Not all graphs are planar. For example, the graph K_5 (consisting of 5 vertices and 10 edges joining every pair of vertices) and the graph K_{33} (consisting of 6 vertices $v_1, v_2, v_3, u_1, u_2, u_3$ and 9 edges joining each vertex v_i to all the vertices u_j) are not planar (this will be proved in the exercise class). If $G \subset \mathbb{R}^2$ is a planar graph, then it divides the plane $\mathbb{R}^2$ into one or more regions, called the faces of the graph. More precisely, a *face* of a planar graph $G \subset \mathbb{R}^2$ is a maximal (by inclusion) arcwise connected subset of $\mathbb{R}^2 \backslash G$.

Let G be a planar graph. Denote by $v(G)$, $e(G)$, and $f(G)$ the number of vertices, edges, and faces of G, respectively.

Theorem 5 [Euler's formula for planar graphs]. *If G is a connected planar graph, then*

$$v(G) - e(G) + f(G) = 2. \tag{1}$$

Sketch of the Proof. We begin with a simple lemma.

Lemma 3. *The edges $e_1, \dots e_k$ of any connected graph G can be numbered so that $G_l := G \backslash \{e_1 \cup \dots \cup e_l\}$ is connected for any $l \leqslant k$.*

The proof is an exercise.

Let us number the edges of the given planar connected graph as in Lemma 3. We will prove the theorem by induction on the number of edges k. If $k = 0$, then $e(G) = 0$, $v(G) = 1$,f(G)=1, and the formula holds. Suppose the formula holds for $k = l$, i.e.,

$$v(G_l) - e(G_l) + f(G_l) = 2; \tag{2}$$

let us prove it for $k = l + 1$.

Consider the edge e_{l+1}. There are two possibilities:

(a) e_{l+1} has one common vertex with G_l;
(b) e_{l+1} has two common vertices with G_l.

In case (a), the number of faces does not change, one vertex and one edge are added. So formula (2) holds for G_{l+1}.

In case (b), a new circuit appears, and so the number of faces increases by one (by the Jordan Curve Theorem), the number of vertices does not change, and one edge is added. Again formula (2) holds for G_{l+1}. Formula (1) is proved. $\square$

Theorem 6 [Kuratowski]. *G is a planar graph if and only if it does not contain as subgraphs neither K_5 nor K_{33}.*

We will prove this theorem in the exercise class.

Lecture 3 TOPOLOGICAL CONSTRUCTIONS

In this lecture we study constructions used in topology. These constructions transform one (or two) given topological spaces into a new topological space.

§1. Cone and Suspension

Roughly speaking, the cone over a space is obtained by joining a fixed point by line segments with all points of the space. More precisely, let $X \subset \mathbb{R}^n$, where $\mathbb{R}^n$ is n-dimensional space with coordinates $x_1, x_2, \dots, x_n$. Let $\mathbb{R}^{n+1}$ be $(n+1)$-dimensional space with coordinates $x_1, x_2, \dots, x_n, x_{n+1}$. Then $X \subset \mathbb{R}^n \subset R^{n+1}$. Let $p \in \mathbb{R}^{n+1}$ be the point with coordinates $(0, 0, \dots, 0, 1)$. Then the *cone over* X is the set

$$C(X) := \bigcup_{x \in X} [p, x],$$

where $[p, x]$ is the line segment joining p and x. The point p is called the *vertex* of the cone. By definition, the cone over the empty set is one point.

Roughly speaking, the suspension over a topological space is obtained by joining two fixed points by segments with all the points of the given space. More precisely, let $q \in \mathbb{R}^{n+1}$ be the point with coordinates $(0, 0, \dots, 0, -1)$. Then the *suspension over* X is the set

$$\Sigma(X) := \bigcup_{x \in X} \left([p, x] \bigcup [x, q]\right),$$

where $[p, x]$ and $[x, q]$ are the line segments joining the corresponding points. By definition, the suspension over the empty set is the two-point set S^0.

Let us define two standard objects: the *n-disk* $D^n := \{x \in \mathbb{R}^n : |x| \leqslant 1\}$ and the *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}$; here $|x|$ denotes the length of the vector $x \in \mathbb{R}^n$.

Proposition 1. *The cone over the n-sphere is the $(n+1)$-disk and the cone over the n-disk is the $(n+1)$-disk. The suspension over the n-sphere is the $(n+1)$-sphere and the suspension over the n-disk is the $(n+1)$-disk.*

The proof is not difficult: in each case the corresponding homeomorphism is easily constructed. $\square$

§2. Cartesian Product and Join

Roughly speaking, the Cartesian product of two spaces is obtained by putting a copy of one of the spaces at each point of the other space. More precisely, let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$. Let $x_1, \dots, x_k$ be the coordinates in $\mathbb{R}^k$; denote the coordinates in $\mathbb{R}^l$ by $x_{k+1}, \dots, x_{k+l}$; then

$$X \subset \mathbb{R}^k \subset \mathbb{R}^{k+l} \supset \mathbb{R}^l \supset Y.$$

The *Cartesian product* of X and Y is

$$X \times Y := \{z \in \mathbb{R}^{k+l} \mid z = xy \; x \in X, y \in Y\};$$

here xy denotes the concatenation of x and y, i.e., the point with coordinates $(x_1, \dots, x_k, x_{k+1}, \dots, x_{k+l})$, where $x = (x_1, \dots, x_k)$ and $y = (x_{k+1}, \dots, x_{k+l})$.

Roughly speaking, the join of two spaces is obtained by joining each pair of points from the two spaces by a segment. More precisely, suppose that $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$. Let $x_1, \dots, x_{k+l+1}$ be the coordinates in $\mathbb{R}^{k+l+1}$. Consider the linear map $\lambda : \mathbb{R}^l \to \mathbb{R}^{k+l+1}$ given by

$$\mathbb{R}^l \ni (x_1, \dots, x_l) \mapsto (0, \dots, 0, 1, x_1, \dots, x_l) \in \mathbb{R}^{k+l+1}.$$

Denote $\mathbb{R}^l_* := \lambda(\mathbb{R}^l)$ and $Y_* := \lambda(Y)$. Then

$$X \subset \mathbb{R}^k \subset \mathbb{R}^{k+l+1} \supset \mathbb{R}^l_* \supset Y_*.$$

The *join* of X and Y is the set $X * Y$ of all points lying on all line segments $[x, y]$, where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^l_*$.

Proposition 2. *The Cartesian product of the n-disk and the m-disk is the $(n+m)$-disk. The join of the n-disk and the m-disk is the $(n+m+1)$-disk. The join of the n-sphere and the m-sphere is the $(n+m+1)$-sphere.*

The proof is not difficult: in each case the corresponding homeomorphism is easily constructed. □

§3*. Quotient spaces

In this section we consider general topological spaces (as in §3, Lecture 1). So this section is optional.

Roughly speaking, a quotient space is obtained from a given space by considering certain subsets of the given space as points ("dividing" our space by these subsets). More precisely, let $(X, \mathcal{T} = \{U_\alpha\})$ be a topological space and $\mathcal{F} = \{\xi_\beta\}$ a family of closed nonintersecting sets in X. The *quotient set* of X by $\mathcal{F}$, denoted $X/_{\mathcal{F}}$, is the set whose "points" are the sets ξ_β and the remaining points of X (i.e., the points of $X - \cup_\beta \xi_\beta$). Let us denote by $\mathcal{T}/_{\mathcal{F}}$ the family of sets $U \subset X/_{\mathcal{F}}$ such that U is an open set from $\mathcal{T}$ if U is regarded as a set in X (i.e. if each "point" ξ_β is considered as a set of points of X rather than a single point of $X/_{\mathcal{F}}$). Then the topological space $(X/_{\mathcal{F}}, \mathcal{T}/_{\mathcal{F}})$ is called the *quotient space* of X by $\mathcal{F}$.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces; the *disjoint union* $X \sqcup Y$ of X and Y is the set of points $\{x \mid x \in X\} \cup \{(y, *) \mid y \in Y\}$ with the topology $\mathcal{T} \cup \mathcal{T}'_*$, where the open sets of $\mathcal{T}'_*$ are all sets of the form $\{(y, *) \mid y \in U' \in \mathcal{T}'\}$. (The star $*$ is needed for the case in which X and Y have common points.)

Suppose X and Y are topological spaces, A and B are closed subspaces of X and Y, respectively, and $f : A \to B$ is a continuous map. In the disjoint union of X and Y, consider the family of sets $\mathcal{F} := \{b \sqcup f^{-1}(b) \mid b \in B\}$. Then we denote the quotient space $(X \cup Y)/_{\mathcal{F}}$ by $X \cup_f Y$ and say that this space is obtained by *attaching* (or *glueing*) Y to X along f.

§4. Cutting and pasting

The operations of cutting and pasting are intuitively very clear for surfaces: you cut the surface along some line with scissors (*cutting*) and you glue together some parts of the edges of the surface (*pasting*).

For the formal definition of cutting, we need a definition. Let A and B be subsets of $\mathbb{R}^n$; suppose there is a parallel translation p in $\mathbb{R}^n$ such that $B' := p(B)$ does not intersect A; then $A \cup B'$ is called the *disjoint union* of A and B and is denoted by $A \sqcup B$.

Suppose a line L on a surface S splits S (i.e., $S \setminus L$ is not arcwise connected and consists of two arcwise connected parts A and B). Then the result of *cutting S along L* is the disjoint union $(A \cup L) \sqcup (B \cup L)$.

The formal definition of pasting of surfaces is a particular case of attaching two spaces by a map (see §3) in which the map is a homeomorphism and the spaces are surfaces. This will be explained in detail in the exercise classes.