Lecture 4 SURFACES

A *closed surface* is a compact space M each point of which has a neighborhood homeomorphic to the open 2-disk Int D^2. A *surface-with-holes* is a compact space N each point of which has a neighborhood homeomorphic to the open 2-disk Int D^2 or to the open half disk Int $D^2_{1/2} = \{(x,y) \in \mathbb{R}^2 | x \geqslant 0, x^2+y^2<1\}$. In this lecture we shall classify closed surfaces and surfaces-with-holes.

§1. PL-surfaces

A (finite) 2-*polyhedron* is a space $X \subset \mathbb{R}^n$ represented as the (finite) union of triangles (its *2-simplices* or *faces*) so that the intersection of two triangles is either empty, or a common side, or a common vertex. The sides of the triangles are called *edges* or 1-*simplices*, the vertices of the triangles are called *vertices* or 0-*simplices* of the 2-polyhedron.

Let P be a 2-polyhedron and $v \in P$ be a vertex. The (closed) *star* of v in P (notation Star(v,P)) is the set of all triangles with vertex v. The *link* of v in P (notation Link(v,P)) is the set of sides opposite to v in the triangles containing v.

A finite 2-polyhedron is said to be a *closed PL-surface* (or a *closed triangulated surface*) if the star of any vertex v is homeomorphic to the closed 2-disk with v at the center (or, which is the same, if the links of all its vertices are homeomorphic to the circle). A finite 2-polyhedron is said to be a *PL-surface-with-holes* if the star of any vertex v is homeomorphic either to the closed 2-disk with v at the center or to the closed disk with v on the boundary (or, which is the same, if the links of all its vertices are homeomorphic either to the circle or to the line segment). It is easy to see that in a PL-surface-with-holes the points whose links are segments (they are called *boundary points*) constitute a finite number of circles (called *boundary circles*). It is also easy to see that each edge of a closed PL-surface (and each nonboundary edge of a surface-with-holes) is contained in exactly two faces.

A PL-surface (closed or with holes) is called *connected* if any two vertices can be joined by a sequence of edges (each edge has a common vertex with the previous one). Further, unless otherwise stated, we consider only connected PL-surfaces.

A PL-surface (closed or with holes) is called *orientable* if its faces can be coherently oriented; this means that each face can be oriented (i.e., a cyclic order of its vertices chosen) so that each edge inherits opposite orientations from the orientations of the two faces containing this edge. An *orientation* of an orientable surface is a choice of a coherent orientation of its faces; it is easy to see that that any orientable (connected!) surface has exactly two orientations.

§2. Subdivisions and PL-homeomorphisms

A *face subdivision* is the replacement of a face (triangle) by three new faces obtained by joining the baricenter of the triangle with its vertices. An *edge subdivision* is the replacement of the two faces (triangles) containing an edge by four new faces obtained by joining the midpoint of the edge with the two opposite vertices of the two triangles. A *baricentric subdivision* of a face is the replacement of a face (triangle) by six new faces obtained by constructing the three medians of the triangles. A *baricentric subdivision of a surface* is the result of the baricentric subdivision of

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

all its faces. Clearly, any baricentric subdivision can be obtained by means of a finite number od edge and face subdivisions. A *subdivision* of a PL-surface is the result of a finite number of edge and face subdivisions.

Two PL-surfaces M_1 and M_2 are called *isomorphic* if there exists a homeomorphism $h : M_1 \to M_2$ such that each face of M_1 is mapped onto a face of M_2. Two PL-surfaces M_1 and M_2 are called *PL-homeomorphic* if they have isomorphic subdivisions.

At the beginning of this lecture, we defined surfaces without any triangulation (without PL-structure). The following theorem says that there is no loss of generality if we consider only PL-surfaces.

PL-Structure Theorem. *Any surface has a triangulation (is a PL-surface). Two surfaces M_1 M_2 are homeomorphic if and only if M_1 and M_2 have PL-homeomorphic triangulations (and in that case any two triangulations of M_1 and M_2 are PL-homeomorphic).*

The proof of this theorem (based on the Jordan Curve Theorem) is technical and will not be given in this course.

From now on we shall often omit the prefix "PL-" when talking about surfaces. If you don't like this (because the PL-structure theorem has not been proved), you can add this prefix in all the appropriate places.

§3. The Euler characteristic of PL-surfaces

The *Euler characteristic* $\chi(M)$ of a PL-surface is defined by

$$\boxed{\chi(M) := V - E + F},$$

where V, E, and F are the numbers of vertices, edges, and faces of M, respectively.

Theorem 1. *The Euler characteristic of a surface does not depend on its triangulation. PL-homeomorphic PL-surfaces have the same Euler characteristic.*

Proof. It follows from the definitions that we must only prove that the Euler characteristic does not change under subdivision, i.e., under face and edge subdivision. But these two facts are proved by a straighrforward verification. $\square$

§3. Classification of surfaces

The *connected sum* $M_1 \# M_2$ of two surfaces M_1 and M_2 is obtained by making two small holes in the surfaces and gluing them along the boundaries of the holes.

Theorem 2. *Any closed orientable surface is homeomorphic to one of the surfaces in the following list*

$$S^2 \text{ (sphere)},\ S^2 \times S^2 \text{ (torus)},\ (S^2 \times S^2)\#(S^2 \times S^2) \text{ (sphere with two handles)}, \dots$$
$$(S^2 \times S^2)\#(S^2 \times S^2)\# \dots \#(S^2 \times S^2) \text{ (sphere with k handles)}, \dots$$

Any two surfaces in the list are not homeomorphic.

Sketch of the proof. First we assume that M is triangulated and take the double baricentric subdivision M'' of M. In this triangulation, the star of a vertex of M''

is called a *cap*, the union of all faces of M'' intersecting an edge of M but not contained in the caps is called a *strip*, and the connected components of the union of the remaining faces of M'' are called *patches*.

Consider the union of all the edges of M; this union is a graph (denoted G). Let G_0 be a maximal tree of G. Denote by M_0 the union of all caps and strips surrounding G_0. Clearly M_0 is homeomorphic to the 2-disk (why?). If we successively add the strips and patches from $M - M_0$ to M_0, obtaining an increasing sequence

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_p = M,$$

we shall recover M.

What happens when we add a strip at the ith step? Two cases are possible: either the number of holes of M_{i-1} increases by 1 or this number does not change. It is not difficult to see that in the first case "inverted pants" are attached to M_{i-1} and in the second case "upright pants" are added to M_{i-1}.

After we have added all the strips, what will happen when we add the patches? The addition of each patch will "close" a pair of pants either at the "legs" or at the "waist". As the result, we obtain a sphere with k handles, $k \geqslant 0$. This proves the first part of the theorem.

To prove the second part, it suffices to compute the Euler characteristic (for some specific triangulation) of each entry in the list of surfaces (obtaining $2, 0, -2, -4, \ldots$, respectively) and apply Theorem 1. $\square$

The statement and proof of similar theorems about nonorientable closed surfaces, orientable surfaces-with-holes, and nonorientable surfaces-with-holes are left as exercises.