Lecture 5 CONFIGURATION SPACES

In this lecture we consider topological spaces which are not defined as subsets of Euclidean space $\mathbb{R}^n$. The elements of such spaces may be of arbitrary nature; what is important is that the family of all open sets must be appropriately defined. When this is the case, all the main definitions from the previous lectures (closure, boundary, interior, compactness, arcwise connectedness, continuous map, homeomorphism, etc.) remain valid, as well as the main theorems. The principal examples considered in this lecture come from classical mechanics; in these examples "points" (i.e., elements) of the spaces under consideration are "positions of a mechanical system", and the spaces themselves are called *configuration spaces*.

## §1. Base of a Topology

A topological space is a set $X$ of elements, usually called *points*, together with a family $\tau$ of subsets of $X$, called *open sets*; the open sets must satisfy certain axioms (they are listed in §2* of Lecture 1, but are not important for us here); the family $\tau$ of open sets is often called the *topology* on $X$. In many cases, instead of describing all the open sets of $\tau$, it is more economical to describe a smaller subfamily $\tau_0 \subset \tau$ from which the whole family $\tau$ can be reconstructed by using only the union operation. The precise definition is the following.

Let $\tau$ be the topology of the space $X$. A subfamily $\tau_0 \subset \tau$ is called a *base* of the topology of $X$ if for any open set $U \in \tau$ there exists a collection of open sets belonging to $\tau_0$ whose union is the set $U$. In particular, a base of a given topology can be described by specifying, for each point of the space, what the "small neighborhoods" of this point are.

For example, a base of the real line $\mathbb{R}$ (respectively of the space $\mathbb{R}^n$) is the set of all open rational intervals $(p, q)$, $p, q \in \mathbb{Q}$, of length less than $1/1000$ (resp., the set of all balls of rational radius $r < 1/1000$ centered at rational points). Note that this base is indeed economical: it is countable (=счетная), while the topology of $\mathbb{R}$ is an uncountable family of sets. A less economical base for $\mathbb{R}^n$ consists of all $\varepsilon$-balls, $\varepsilon < 1/100$, in $\mathbb{R}^n$.

## §2. Examples of configuration spaces

A *mechanical system* in classical mechanics is a mathematical model of a real mechanical device that works according to the classical laws of physics. An example of a mechanical system is the set of $n$ distinct massive point particles in $\mathbb{R}^3$ interacting according to Newton's laws; another example is the planar *pendulum*, a massive segment fixed at one end by a hinge (=шарнир) that allows it to oscillate in the vertical plane under the action of the force of gravity. Here we will not be interested in the specific laws of physics acting on our system; we are only concerned with the geometry of the various *positions* that the system can assume. The set of all positions of a given mechanical system is said to be the *configuration space* of the system; this set is usually supplied with the "natural topology", and so it is a topological space. As we shall see in the examples below, the natural topology is usually specified by indicating what a "small neighborhood" of each position is.

Thus in the $n$-particle example a typical small neighborhood of a position $p_0$ of the system can be specified by the union of all open balls of radius $\varepsilon$ centered at the particles: another position $p$ belongs to the $\varepsilon$-neighborhood of $p_0$ if each particle of $p$ lies in the corresponding $\varepsilon$-ball; in this model the configuration space is obviously homeomorphic to $\mathbb{R}^{3n}$. A more realistic $n$-particle model arises if two particles are not allowed to coincide (occupy the same position in space); the configuration space is then $\mathbb{R}^{3n}$ from which the "diagonal", i.e., the set

$$D := \{(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n) \in \mathbb{R}^{3n} | \quad x_i = x_j, \\ y_i = y_j, \ z_i = z_j, \ \ i \neq j, \ i, j \in \{1, \dots, n\}\},$$

is removed.

In the case of the pendulum, the natural topology can be defined by using the notion of $\varepsilon$-angle around a position of the segment, and of course the configuration space is the circle $S^1$.

As a final example, let us consider a two-sided symmetric cone with vertex angle $73°$ in 3-space and a fixed hinge at the vertex that allows the double cone to rotate freely in space. Then each position of the cone is specified by its axis of symmetry and the natural topology can be introduced by using the notion of $\varepsilon$-cone (do it!). It is easy to see (how?) that the configuration space of this mechanical system is the projective plane $P\mathbb{R}^2$.

## §3. Planar hinge mechanisms

In this section we consider mechanical systems consisting of rods (=стержни) joined by hinges (=шарниры) around which the rods can rotate. More precisely, a *planar hinge mechanism* is a mobile planar graph whose edges (called *rods*) are line segments of fixed lengths and whose vertices (called *hinges*) are either fixed points of the plane or mobile points that can change their position in the plane inasmuch as the (rigid) rods and fixed hinges allow them to. The set of all positions of a hinge mechanism is called its *configuration space*; the topology in this space is defined in the natural way (namely, a base of the topology consists, for each position $x_0$ of the mechanism, of all positions such that each hinge is displaced from its position in $x_0$ by less than $\varepsilon$, where (say) $\varepsilon < 1/1700$).

It is easy to see that the configuration space of the *pendulum* (one rod with one fixed hinge) is the circle, while the configuration space of the *double pendulum* (two rods joined by a mobile hinge, one other endpoint being a fixed hinge) is the torus $T^2 = S^1 \times S^1$.

Let us define some special classes of hinge mechanisms and introduce some notation for them. A *quadrangle* is a planar hinge mechanism with 4 hinges and 3 rods: the first hinge is fixed, the second hinge is mobile and joins the first rod to the second one, the third hinge is also mobile and joins the second and third rod, the third rod's other endpoint is the fourth hinge, which is fixed; a quadrangle is denoted by $\langle l_1, l_2, l_3; d\rangle$, where $l_1, l_2, l_3$ are the lengths of the successive rods and $d$ is the distance between the fixed hinges. The *pentagon* $\langle l_1, l_2, l_3, l_4; d\rangle$ and more generally the *n-gon* $\langle l_1, l_2, \dots, l_{n-1}; d\rangle$ is defined similarily.

Here are some examples that will be discussed in the exercise class.

(i) The "square" $\langle 1, 1, 1; 1\rangle$ has a configuration space homeomorphic to three circles each pair of which are tangent (i.e., have one common point).

(ii) The "thin pentagon" $\langle 1,1,1,1;3.9\rangle$ has a configuration space homeomorphic to the 2-sphere.

(iii) The "high pentagon" $\langle 4,1,1,4;1\rangle$ has a configuration space homeomorphic to the disjoint union of two tori.

(iv)* The "regular pentagon" $\langle 1,1,1,1;1\rangle$ has a configuration space homeomorphic to the 2-sphere with four (!) handles.

## §4. Generic Pentagons

A pentagon is called *nongeneric* (=не типичным) if there exists a position in which all the hinges lie on one straight line; otherwise it is called *generic*. For example, the thin pentagon is generic, but the regular pentagon is not.

It is easy to see that the configuration space of any pentagon is a subset of $\mathbb{R}^6$. Indeed, if we denote by $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ the coordinates (in the plane $\mathbb{R}^2$) of the three mobile hinges for some position $p$ of the pentagon, and by $(x_0,y_0),(x_4,y_4)$ the fixed hinges, the correspondence

$$p=((x_1,y_1),(x_2,y_2),(x_3,y_3))\mapsto(x_1,y_1,x_2,y_2,x_3,y_3)\in\mathbb{R}^6$$

maps the set of positions of our pentagon bijectively onto the subset of $\mathbb{R}^6$ given by the system of equations

$$\begin{gathered}\sqrt{(x_0-x_1)^2+(y_0-y_1)^2}=l_1,\quad \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=l_2,\\ \sqrt{(x_2-x_3)^2+(y_2-y_3)^2}=l_3,\quad \sqrt{(x_3-x_4)^2+(y_3-y_4)^2}=l_4,\\ \sqrt{(x_0-x_4)^2+(y_0-y_4)^2}=d.\end{gathered}\tag{1}$$

**Theorem 1.** *The configuration space of a generic pentagon does not depend on the order in which the rods are attached to each other, i.e., on the order in the set of lengths $\{l_1,l_2,l_3,l_4,d\}$, it depends only on the (unordered) set itself.*

**Proof.** The theorem immediately follows from the above, because we can change the order of the coordinates in $\mathbb{R}^6$ without affecting system (1), which defines the configuration space in $\mathbb{R}^6$. $\square$

For generic pentagons there is a nice classification theorem.

**Theorem 2.** *The configuration space of a generic pentagon is either the sphere with $k$ handles, $k=0,1,2,3,4$, or the disjoint union of two tori.*

The proof of this theorem will be discussed in the exercise class.