Lecture 6 VECTOR FIELDS ON THE PLANE

The notion of vector field comes from mechanics and physics. Examples: the velocity field of the particles of a moving liquid in hydrodynamics, or the field of gravitational forces in Newtonian mechanics, or the field of electromagnetic induction in electrodynamics. In all these cases, a vector is given at each point of some domain in space, and this vector changes continuously as we move from point to point. In this lecture we will study a simpler model situation: vector fields on the plane (rather than in space).

§1. Trajectories and singular points

A *vector field* V in the plane $\mathbb{R}^2$ is a rule that assigns to each point $p \in \mathbb{R}^2$ a vector $V(p)$ issuing from p. Such an assignment may be expressed in the coordinates x, y of $\mathbb{R}^2$ as

$$X = \alpha(x, y) \quad Y = \beta(x, y),$$

where $\alpha : \mathbb{R}^2 \to \mathbb{R}$ and $\beta : \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions on the plane, (x, y) are the coordinates of the point p, and (X, Y) are the coordinates of the vector $V(p)$. If the functions α and β are continuous, then the vector field V is called *continuous*.

A *trajectory* through the point $p \in \mathbb{R}^2$ is a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ passing through p and tangent at all its points to the vector field (i.e., the vector $V(q)$ is tangent to the curve $C := \gamma(\mathbb{R})$ at each point $q \in C$). A *singular point* p of a vector field V is a point where V vanishes: $V(p) = 0$; when V is a velocity field, such a point is often called a *rest point*, when V is a field of forces, it is called an *equilibrium point.*

§2. Generic singular points of plane vector fields

We will now define certain types of singular points of plane vector fields. To define these points, we will not write explicit formulas for the vectors of the field, but instead describe the topological picture of its trajectories near the singular point and give physical examples of such singularities.

The *node* (=узел) is a singular point contained in all the nearby trajectories; if all the trajectories move towards the point, the node is called *stable* and *unstable* if all the trajectories move away from the point. As an example, we can consider the gravitational force field of water droplets flowing down the surface $z = x^2 + y^2$ near the point $(0, 0, 0)$ (stable node) or down the surface $z = -x^2 - y^2$ near the same point (unstable node).

The *saddle* (=седло) is a singular point contained in two transversal trajectories, called *separatices*, one of which is ingoing, the other outgoing, the other trajectories behaving like a family of hyperbolas whose asymptotes are the separatrices. As an example, we can consider the gravitational force field of water droplets flowing down the surface $z = x^2 - y^2$ near the point $(0, 0, 0)$; here the separatrices are the coordinates axes.

The *center* is a singular point near which the trajectories behave like the family of concentric circles centered at that point; a center is called *positive* if the trajectories rotate counterclockwise and *negative* if they rotate clockwise. As an example, we can consider the velocity field obtained by rotating the plane about the origin with constant angular velocity.

The *focus* is a singular point that ressembles the node, except that the trajectories, instead of behaving like the set of straight lines passing through the point, behave as a family of logarithmic spirals converging to it (*stable focus*) or diverging from it (*unstable focus*).

FIGURE 1. Generic singular points

A singular point is called *generic* if it is of one of the four types described above (node, center, saddle, focus). A vector field is called *generic* if it has a finite number of singular points all of which are generic. In what follows we will only consider generic vector fields.

Remark 1. Let us explain informally why the term generic is used here. Generic fields are, in fact, the "most general" ones in the sense that, first, they occur "most often" (i.e., as close as we like to any vector field there is a generic one) and, second, they are "stable" (any vector field close enough to a generic one is also generic, has the same number of singular points, and those points are of the same types). These statements are not needed in this course, so we will not make them more precise nor prove them.

Remark 2. It can be proved that the saddle and the center are *not* topologically equivalent to each other and not equivalent to the node or to the focus; however, the focus and the node *are* topologically equivalent; as topologists, we should not distiguish them, but we do, following the traditions of the theory of dynamical systems (where an equivalence relation stricter than homeomorphism is used). We do not use (an hence do not define) this relation.

§3. The index of plane vector fields

Suppose a vector field V in the plane is given. Let $\gamma : S^1 \to \mathbb{R}^2$ be a closed curve in the plane not passing through any singular points of V; denote $C := \gamma(S^1)$. To each vector $V(c)$, $c \in C$, let us assign the unit vector of the same direction as $V(c)$ issuing from the origin of coordinates $O \in \mathbb{R}^2$; we then obtain a map $g : C \to S^1_1$

(where $S_1^1 \subset \mathbb{R}^2$ denotes the unit circle centered at O), called the *Gauss map* corresponding to the vector field V and to the curve γ. Let the point $c \in C$ go once around the curve C; then the endpoint of the unit vector $V(c)/|V(c)|$ moves along the circle S_0^1, finally returning to its starting position; the total number of revolutions in the positive (counterclockwise) direction that it performs is called the *index of the vector field V along the curve γ* and is denoted by $\mathrm{Ind}(\gamma, V)$, or by $\mathrm{Ind}(\gamma)$ if the choice of V is clear from the context.

Remark 3. A simple way of computing $\mathrm{Ind}(\gamma)$ is to fix a ray issuing from O (say the half-axis Ox) and count the number of times p the endpoint of $V(c)$ passes through the ray in the positive direction and the number of times q in the negative one; then $\mathrm{Ind}(\gamma) = p - q$.

Theorem 1. *Suppose that a simple closed curve γ does not pass through any singular points of a vector field V and bounds a domain that also does not contain any singular points of V. Then*

$$\boxed{\mathrm{Ind}(\gamma, V) = 0\ .}$$

Proof. By the Jordan curve theorem, we can assume that there exists a homeomorphism of $\mathbb{R}^2$ that takes the domain bounded by $C := \gamma(S^1)$ to the unit disk centered at the origin O. This homeomorphism maps the vector field V to a vector field that we denote by V'. Obviously,

$$\mathrm{Ind}(\gamma, V) = \mathrm{Ind}(S_0^1, V), \tag{1}$$

where S_1^1 denotes the unit circle centered at O. Consider the family of all circles S_r^1 of radius $r < 1$ centered at O. The vector $V'(O)$ is nonzero, hence for a small enough r_0 all the vectors $V'(s)$, $s \in S_{r_0}^1$, differ little in direction from $V'(O)$, so that $\mathrm{Ind}(S_r^1, V) = 0$. But then by continuity $\mathrm{Ind}(S_r^1, V) = 0$ for all $r \leqslant 1$. Now the theorem follows from (1). □

Now suppose that V is a generic plane vector field and p is a singular point of V. Let C be a circle centered at p such that no other singular points are contained in the disk bounded by C. Then the *index of V at the singular point p* is defined as $\mathrm{Ind}(p, V) := \mathrm{Ind}(C, V)$. This index is well defined, i.e., it does not depend on the radius of the circle C (provided that the disk bounded by C does not contain any other singular points); this follows from the next theorem.

Theorem 2. *Suppose that a simple closed curve γ does not pass through any singular points of a vector field V and bounds a domain that contains exactly one singular point a of V. Then*

$$\boxed{\mathrm{Ind}(\gamma, V) = \mathrm{Ind}(a, V)\ .}$$

The proof is similar to that of Theorem 1 and is left as an exercise. □