VECTOR FIELDS ON SURFACES

In this lecture we discuss vector fields on orientable surfaces. Here we will see that there is a deep relationship between the *global* topological properties of the surface and the structure of vector fields on it. The previous lecture will serve as the *local* version of the theory.

§1. What is a vector field on a surface?

The simplest nontrivial example of a vector field on a surface is the velocity field of a configuration space $\mathrm{Conf}(M)$ of a mechanism M when $\mathrm{Conf}(M)$ is indeed a surface (as in the case of generic pentagons, see Lecture 5). In order to define this notion mathematically, we will assume that our (compact closed orientable) surface M has a tangent plane at each point p; we denote this plane by $T_p(M)$. Then a vector field V is an assignement of a vector $V(p) \in T_p(M)$ to each point $p \in M$. In a disk neighborhood D of any point $p \in M$ we can identify the tangent plane $T_p(M)$ with the standard coordinate plane $\mathbb{R}^2$, and think of all the other tangent planes $T_q(M)$, $q \in D$, as other copies of $\mathbb{R}^2$, only shifted to the point q.

Thus *locally* the situation here is the same as in the previous lecture: one can define the continuity of vector fields, trajectories, singular points of a vector field, generic vector fields, the index of a vector field at a singular point, etc. However, *for an arbitrary curve $\gamma : S^1 \to M$, the index of a vector field cannot be correctly defined*, because the Gauss map (see §3 of Lecture 6) uses the parallel shift of vectors to a common origin, and such a shift is not well defined on an arbitrary surface. Nevertheless, Theorems 1 and 2 of the previous lecture remain valid *provided* that they are understood locally, i.e. as taking place in a disk $D \subset M$.

As for the case of a plane, a *generic vector field on a surface M* is defined as a continuous vector field on M with a finite number of singular points, all of which are generic (i.e., are nodes, or centers, or foci, or saddles).

§2. Two lemmas

The two following lemmas will be needed in the proof of the main result of this lecture, the Poincaré theorem.

Lemma 1. *If p is a nonsingular point of a vector field V, D is a disk centered at p, and V_0 is any nonzero vector, then there exists another vector field W with the same index, coinciding with V outside of D and such that $W(p) = V_0$.*

Proof. By continuity, there is a disk D_0 concentric to D such that all the vectors $V(q)$, $q \in D_0$, have a direction that differs by less than $1°$ from the direction of $V(p)$. Let r be the radius of D_0, α be the angle between $V(p)$ and V_0, and S^1_s be the circle of radius $s \leqslant r$ centered at p. Then the required vector field W is obtained from V by rotating all the vectors $V(m)$, $m \in S^1_s$, by the angle $\alpha \cdot (r-s)/r$ and replacing $V(p)$ by V_0. $\square$

Lemma 2. *For any generic vector field on a surface there exists a triangulation of the surface such that no singular point is a vertex or a point of an edge of the triangulation, and each open face contains no more than one singular point.*

Proof. First, by obvious modifications of the triangulation, we insure that no singular points lie at vertices or on edges of the triangulation. Then we appropriately subdivide those faces that contain more than one singular point. □

§3. Poincaré's index theorem

Henri Poincaré proved the following beautiful theorem, establishing a deep connection between the character of singular points of vector fields and the topology (as expressed by the Euler characteristic) of the surface on which they are defined.

Theorem. *The index of any generic vector field on a (closed compact connected triangulated) orientable surface is equal to the Euler characteristic of this surface.*

Proof. The proof will be in two parts. In the first part, we will construct a special vector field whose index is indeed equal to the Euler characteristic of the surface. In the second part, we will prove that all generic vector fields on a given surface have the same index.

Part 1. Let us fix a triangulation of our surface. We will construct a special continuous vector field on the triangulated surface with singular points at all the vertices, at the midpoints of all the edges, and at the baricenters of all the faces. At the midpoint of each edge, we place a saddle point whose ingoing separatrix goes along the edge and whose outgoing separatrix goes to the baricenters of the two triangular faces adjacent to the edge. At each vertex, we place an unstable node so that the edges issuing from the vertex are covered by outgoing trajectories of the node. At the baricenter of each face, we place a stable node so that its ingoing trajectories include the three separatrices coming to the baricenter from the three saddle points at the midpoints of the face's three sides. Finally, it is easy to see that the vector fields already constructed in the neighborhoods of the three types of points (vertices, midpoints, baricenters) can be extended continuously so as to cover the entire surface.

Figure 1. The special vector field

The index of the vector field thus constructed is obviously equal to the Euler characteristic $\chi = V - E + F$ of the surface. Indeed, the nodes at the vertices and the baricenters have index equal to $+1$, so that the nodes contribute $V + F$ to the index, while the saddle points have index equal to -1, so they contribute $-E$, and all that adds up to χ.

Part 2. Let V_1 and V_2 be two generic vector fields on our surface; our aim is to prove that they have the same index. First, by using Lemma 2, we can assume that all the singular points of V_1 and V_2 lie inside the faces (triangles) of the triangulation, no more than one in each face. Next, by applying Lemma 1 at each vertex, we can assume that the vectors $V_1(a)$ and $V_2(a)$ have the same direction at each vertex a.

Now let us fix an orientation of M. Then each edge ab acquires two opposite orientations, ab and ba, from the two faces adjacent to it. Let a mobile point x move from a to b and then back to a; as x moves from a to b, consider the rotation of the vector issuing from a and equal to $V_1(x)$ followed by the rotation of the vector issuing from a and equal to $V_2(x)$ as the point x moves back from b to a; denote by d_{ab} the number of revolutions performed by the vector (d_{ab} is a well-defined integer, because the two vector fields coincide at the vertices). In a similar way, we can define d_{ba}. Obviously, $d_{ab} = -d_{ba}$. Summing over the set $\mathcal{E}$ of all edges, we can write

$$\sum_{(ab)\in\mathcal{E}} (d_{ab} + d_{ba}) = 0. \tag{2}$$

Now let us look at this sum from the point of view of the set $\mathcal{F}$ of faces. Let $(abc) \in \mathcal{F}$, where the cyclic order a, b, c agrees with the chosen orientation of M. Now consider the sum $d_{ab} + d_{bc} + d_{ca}$; it does not change if we first perform the rotation of all the vectors V_1 and then of all the vectors V_2; hence by applying Theorem 2 twice, we obtain

$$d_{ab}+d_{bc}+d_{ca} = \mathrm{Ind}(\langle abc\rangle, V_1)+\mathrm{Ind}(\langle bac\rangle, V_2) = \mathrm{Ind}(\langle abc\rangle, V_1)-\mathrm{Ind}(\langle abc\rangle, V_2), \tag{3}$$

where $\langle abc\rangle$ denotes the (positively oriented) closed curve bounding the face (abc). Rewriting the sum (2) as a sum over the faces and using (3), we obtain:

$$0 = \sum_{(abc)\in\mathcal{F}} (\mathrm{Ind}(\langle abc\rangle, V_1) - \mathrm{Ind}(\langle bac\rangle, V_2)) =$$
$$\sum_{(abc)\in\mathcal{F}} (\mathrm{Ind}(\langle abc\rangle, V_1) - \sum_{(abc)\in\mathcal{F}} (\mathrm{Ind}(\langle abc\rangle, V_2))) = \mathrm{Ind}(M, V_1) - \mathrm{Ind}(M, V_2)).$$

The theorem is proved. □

§4. Applications

Here we state only two immediate applications of Poincaré's theorem (there will be more in the exercise classes).

Corollary 1. *Any generic vector field on the sphere has at least two singular points.*

Corollary 2. *Any continuous force field on the configuration space of the pentagon* $\langle 1, 1, 1, 1; 3.9\rangle$ *has at least two equilibrium points.*