INFINITE CONSTRUCTIONS

The aim of this lecture is to demonstrate that many "obvious" topological statements turn out to be false. This is done by providing the appropriate counterexamples, all of which, in their day, were little mathematical sensations. All the examples described in this lecture are the result of infinitely iterated geometric constructions.

§0. Measure, dimension, simple connectedness

This section contains three important definitions dealing with certain basic characteristics of sets.

The measure of a set $A \subset \mathbb{R}$ is the generalization of the notion of length of an interval $[a,b] \subset \mathbb{R}$. It is defined axiomatically: the *measure* on $\mathbb{R}$ is a function μ (with values in $\mathbb{R} \cup \{+\infty\}$) defined on a certain family $\mathcal{M}$ of subsets of $\mathbb{R}$, called *measurable*, and satisfying the following conditions:

(I) *Normalization*: $\mu([0,1]) = 1$.

(II) *Countable additivity*: For any countable subfamily $\{A_i\} = \mathcal{A} \subset \mathcal{M}$ of pairwise nonintersecting sets, we have

$$\mu\left(\cup_i A_i\right) = \sum_i \mu(A_i).$$

(III) *Complementarity*: For any $A, B \in \mathcal{M}$, $B \supset A$, we have

$$\mu(B \backslash A) = \mu(B) - \mu(A).$$

(IV) *Invariance*: For any isometry $h : \mathbb{R} \to \mathbb{R}$ and any $A \in \mathcal{M}$, we have

$$\mu(h(A)) = \mu(A).$$

Note that not all subsets of $\mathbb{R}$ are measurable, and some have infinite measure, e.g. $\mu(\mathbb{R}) = +\infty$.

The notion of dimension formalizes the idea that the line is one-dimensional, or, say, space is three-dimensional. One of the possible definitions of the dimension of a space $X \subset \mathbb{R}^N$ is given by induction: the (*inductive*) *dimension* of the empty set is -1, $\dim(\varnothing) = -1$; the set X is of dimension $\leqslant d$ at the point $x \in X$, $\dim(X, x) \leqslant d$, if the point x has a neighborhood U, $x \in U \subset X$, whose boundary ∂U in X is of dimension $\leqslant d-1$ at all points; the set X is of dimension d, $\dim(X, x) = d$, if it is of dimension $\leqslant d$ but not of dimension $\leqslant d-1$.

The notion of simple-connectedness formalizes the idea that a space X has no holes, i.e., any curve in it can be continuously pulled to a point. The formal definition is as follows: the space $X \subset \mathbb{R}^N$ is *simply connected* if any continuous map $f : S^1 \to X$ of the circle to X can be extended to the disk D^2 bounded by the circle.

§1. The Cantor set

It follows from the definition of dimension zero that a zero-dimensional subset of the real line $\mathbb{R}$ cannot contain any interval; hence the following statement is obvious:

Typeset by $\mathcal{AMS}$-TEX

"Theorem" 1. *Any measurable set $A \subset \mathbb{R}$ of dimension* 0 *has measure* 0.

However, this statement is false. The counterexamples are known as *Cantor sets.* One of them is obtained in the following way. First we remove the open interval $(1/3, 2/3)$ from the closed interval $[0, 1]$. Then from the remaining intervals $[0, 1/3]$, $[2/3, 1]$ we similarly remove the open intervals $(1/9, 2/9)$, $(7/9, 8/9)$. We continue in the same way, at the nth step removing the central open interval of length $1/3^n$ from each of the remaining closed intervals. If A is the union of all the removed intervals, then the Cantor set is defined as $C_{1/3} := [0, 1]\backslash A$. This version does not provide a counterexample to "Theorem" 1. Indeed, we do have $\dim(C_{1/2}) = 0$, but $\mu(A) = 1$ (this follows from countable additivity by summing the appropriate geometric progression – do it!), so that $\mu(C_{1/3}) = 0$.

But now if we remove not a third of each interval, but say, one ninth, then (again summing the corresponding geometric progression) we obtain $\mu(A) < 1$, so that $\mu(C_{1/9}) \neq 0$. This shows that "Theorem" 1 is false.

§2. Peano's curve

Suppose $f : X \to Y = f(X)$ is a continuous map; then, of course, the dimension of Y can be less than the dimension of X (e.g. if f is a projection). Now suppose X is the closed interval $[0, 1]$; then under f certain points of $[0, 1]$ are identified, so the dimension of Y cannot be more than 1; hence the following statement is obvious:

"Theorem" 2. *The dimension of any curve, i.e., of any continuous image of the interval* $[0, 1]$*, is equal to* 1 (*or to* 0 *in the degenerate case, when this image reduces to one point*). However, this statement is false!

Indeed, we will show that *there exists a continuous map* (*called the Peano curve*) π *of the interval* $I = [0, 1]$ *onto the square* $Q = [0, 1] \times [0, 1]$. To define the map π, we will divide the interval I into smaller intervals and the square Q into smaller squares; these smaller squares and intervals will be supplied with indices in a specific way.

Namely, at the first step we divide I into four equal segments $I^1_1, I^1_2, I^1_3, I^1_4$ numbered from left to right and Q into four equal squares $Q^1_{i_1}$ numbered in the order shown in Fig. 1(a); at the second step we divide I into 16 equal segments $I^2_{i_2}$, numbered from left to right, and Q into 16 equal squares numbered in the order given by the dotted curve in Fig. 1(b);...; at the nth step we divide I into 4^n equal segments $I^n_{i_n}$, numbered from left to right, and Q into 4^n equal squares numbered in the order given by the corresponding curve, obtained by subdividing each square of order $n - 1$ into 4 squares and iterating the construction of the dotted curve (for n=3, this order is shown in Fig. 1(c)).

Now we can define the map π. Let $x \in I$. Then there exists an infinite nested sequence of indices $i_1(x), i_2(x), \ldots, i_n(x), \ldots$ such that

$$I^1_{i_1(x)} \cap I^2_{i_2(x)} \cap \cdots \cap I^n_{i_n(x)} \cap \cdots = x. \tag{1}$$

Let us consider the sequence of squares *with the same indices.* It follows from our construction that $Q^1_{i_1(x)} \subset Q^2_{i_2(x)} \subset \cdots \subset Q^n_{i_n(x)} \subset \ldots$ and since the diameters of these squares tend to zero, there exists a unique point $y \in Q$ such that

$$Q^1_{i_1(x)} \cap Q^2_{i_2(x)} \cap \cdots \cap Q^n_{i_n(x)} \cap \cdots = y. \tag{2}$$

Now we define $\pi(x) := y$. It is easy to see that π is well-defined, i.e., if there is more than one sequence of indices $i_1(x), i_2(x), \dots, i_n(x), \dots$ for which we have (1), we obtain the same point y from (2). The continuity of π is also obvious. To show that $\pi(I) = Q$, note that for any $y \in Q$ there exists a sequence of indices $i_1(y), \dots, i_n(y), \dots$ for which the intersection of all the $Q^n_{i_n(y)}$, $n = 1, 2, \dots$ is y; hence if $x \in I$ is the intersection point of the I's with the same indices, then $\pi(x) = y$. $\square$

FIGURE 1. Approximations to Peano's curve

§3. ANTOINE'S NECKLACE

Consider a zero-dimensional subset A in Euclidean space $\mathbb{R}^3$. Then (by definition of dimension zero) any point $x \in A$ has a neighborhood as small as we wish whose boundary in A is empty, so there are no points that get in the way of pulling any curve outside of A to a point. Hence the following statement is obvious.

"Theorem" 3. *The complement to any zero-dimensional compact subset of Euclidean space $\mathbb{R}^3$ is simply connected.*

But this is false! Indeed, let T_1 be the solid torus in $\mathbb{R}^3$; in it, take a closed chain of four linked solid tori as shown in Fig. 2; denote their union by T_2; in each of these four tori, take a closed chain of four smaller linked solid tori; denote the union of these 16 tori by T_3; continuing in the same way, denote by T_n the union of the 4^n tori obtained at the nth step. By definition, *Antoine's necklace* is the set

$$\mathcal{A} := T_1 \cap T_2 \cap T_3 \cap \dots \cap T_n \cap \dots .$$

The proof of the fact that $\dim(\mathcal{A}) = 0$ is staightforward. Unfortunately, we do not have the necessary techniques to prove that $\mathbb{R}^3 \backslash \mathcal{A}$ is not simply connected; however, it is geometrically clear that the curve C shown in Fig. 2 cannot be pulled to a point. Thus Antoine's necklace invalidates the claim of "Theorem" 3.

FIGURE 2. Antoine's necklace and Alexander's horned sphere

§4. Brouwer's continuum

Consider a geographical map with three countries. Then any two of the countries can have common boundaries consisting of a curve (or several arcs of curves), but the boundaries common to all three consist of isolated points. Hence the following statement is obvious:

"Theorem" 4. *Three countries can have only a finite number of boundary points common to all three: more precisely, three connected sets in the plane whose union is the whole plane and which intersect only in boundary points can have only a finite number of points common to all three.*

However, this statement is false.

To construct the counterexample, we will use a computer program. Consider the complex polynomial $f(z) := z^3 - 1$. Clearly, it has three roots: $1,\, 1/2 + \sqrt{3}/2, /, 1/2 - \sqrt{3}/2$. Let us take an arbitrary point $z_0 \in \mathbb{C}$ and apply the *Newton algorithm* (for the approximate solutions of equations $f(z) = 0$ to this point, i.e, consider the sequence of complex numbers

$$z_0,\, z_1 := z_0 - \frac{f(z_0)}{f'(z_0)}, \dots, z_{n+1} := z_n - \frac{f(z_n)}{f'(z_n)}, \dots .$$

If this sequence converges to the root 1, we call the point z_0 *blue*, if it converges to $1/2 + \sqrt{3}/2$, we call it *red*, and if it converges to $1/2 - \sqrt{3}/2$, we call it *green*. In this way we obtain three "countries" in the complex plane $\mathbb{C}$ such that any point of the boundary of one of the countries is simultaneously a boundary point of the other two!

Another construction (the original one, due to Brouwer) of this common boundary, called the *Brouwer continuum*, will be described in the lecture.

§5. Alexander's horned sphere

The Jordan curve theorem says that the topological circle in the plane bounds a topological disk; its immediate generalization (about the topological sphere in 3-space) is no less obvious and must also be true:

"Theorem" 2. *The topological sphere in space bounds a topological 3-disk, i.e., if $h : S^2 \to \mathbb{R}^3$ is a homeomorphism, then its image, the topological sphere $\Sigma := h(S^2)$ divides $\mathbb{R}^3$ into two components $\mathbb{R}^3 \backslash \Sigma = A \cup B$, the closure of one of which is homeomorphic to the disk D^3.*

However, this statement is false: see Fig. 2(b) or Arkadi's T-shirt.

Lecture 10

HOMOTOPY

The notions of homotopy and homotopy equivalence are quite fundamental in topology. Homotopy equivalence of topological spaces is a weaker equivalence relation than homeomorphism, and *homotopy theory* studies topological spaces up to this relation (and maps up to homotopy). This theory constitutes the main body of *algebraic topology*, but we only consider a few of its basic aspects here.

§1. Homotopic maps

Two maps $f, g : X \to Y$ are called *homotopic* (notation $f \simeq g$) if they can be joined by a *homotopy*, i.e., by a map $F : X \times [0,1] \to Y$ such that $F(x,0) \equiv f(x)$ and $F(x,1) \equiv g(x)$ (here $\equiv$ means for all $x \in X$). If we change the notation from $F(x,t)$ to $F_t(x)$, we can restate the previous definition by saying that there exists a family $\{F_t(x)\}$ of maps, parametrized by $t \in [0,1]$, continously changing from $f \equiv F_0$ to $g \equiv F_1$.

It is easy to prove that

(1) $f \simeq f$ for any $f : X \to Y$ (*reflexivity*);
(2) $f \simeq g \implies f \simeq g$ for all $f, g : X \to Y$ (*symmetry*);
(3) $f \simeq g$ and $g \simeq h \implies f \simeq h$ for all $f, g, h : X \to Y$ (*transitivity*).

For example, to prove transitivity (3), we obtain a homotopy joining f and h by setting

$$F(x,t) = \begin{cases} F_1(x, 2t) & \text{for } 0 \leqslant t \leqslant 1/2 \\ F_2(x, 2t-1) & \text{for } 1/2 \leqslant t \leqslant 1, \end{cases}$$

where F_1, F_2 are homotopies joining f and g, g and h, respectively.

Thus homotopy of maps is an equivalence relation, so that the set $\mathrm{Map}(X,Y)$ of all (continuous) maps of X to Y splits into equivalence classes, called *homotopy classes*; the set of these equivalence classes is denoted $[X,Y]$.

§2. Homotopy equivalence of spaces

Two spaces X and Y are called *homotopy equivalent* if there exist two maps $f : X \to Y$, $g : Y \to X$ (called *homotopy equivalences*) such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Obviously, *homeomorphic spaces are homotopy equivalent* (the homotopy equivalences are provided by any homeomorphism and its inverse). The converse statement is not true: for example, the point is homotopy equivalent to the 2-disk, but these two spaces are not homeomorphic.

Thus homotopy equivalence is a weaker equivalence relation than homeomorphism, so that homotopy classification is rougher (and hence easier – there are less classes) than topological classification. Its importance in topology is due to the fact that most topological invariants are homotopy invariants (this is the case of the so-called fundamental group, homology groups and related invariants such as the Euler characteristic).

§3. Degree of maps of S^1 into itself

In this section we consider (continuous) maps $f : S^1 \to S^1$ of the circle into itself. Examples are the maps $w_k : S^1 \to S^1$ given by the rule $e^{i\varphi} \mapsto e^{ik\varphi}$, where S^1 is modeled by unimodular complex numbers: $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We are interested in the homotopy classification of such maps.

Theorem. *There is a natural bijection between homotopy classes of maps of the circle into itself and the integers,*

$$\boxed{[S^1, S^1] \overset{1-1}{\longleftrightarrow} \mathbb{Z}.} \tag{1}$$

Proof. Consider the map $\exp : \mathbb{R} \to S^1$ given by the rule $\mathbb{R} \ni \varphi \mapsto e^{i\varphi} \in S^1$. The map exp is not a bijection; for example, it takes all points of the form $2k\pi$ to $1 \in S^1$ (see Fig. 1(a)). However, exp is a *local homeomorphism*, i.e., any point has a neighborhood U (e.g. any open interval of length less than 2π containing the point) such that the restriction $\exp|_U$ of exp to U is a homeomorphism.

Now any map $S^1 \to S^1$ can be regarded as a map $f : [0,1] \to S^1$ such that $f(0) = f(1) = 1 \in S^1$. For any such map there exists a unique map $\widetilde{f} : [0,1] \to \mathbb{R}$, called the *lift* of f, such that $\exp \circ \widetilde{f} = f$. Indeed, let us take a subdivision of $[0,1]$ into segments $[0, a_1], [a_1, a_2], \ldots, [a_m, 1]$, so small that none of the images of these segments covers S^1; then, using the fact that exp is homeomorphic on intervals shorter than 2π, we first construct the (unique!) lift of $[0, a_1]$ from the point $0 \in \mathbb{R}$, then the lift of $[a_1, a_2]$ from the image of a_1 in the previous lift, and so on. Now we define the *degree* of the map f, denoted $\deg(f)$, as the integer $\widetilde{f}(1)/2\pi$.

Figure 1. The exponential map, liftings, and a homotopy

Now we are ready to prove the theorem, i.e., establish the bijection (1). We do it by assigning $\deg(f)$ to every homotopy class $[f] \in [S^1, S^1]$. To prove that this assignement is well defined (i.e., does not depend on the choice $f \in [f]$) and bijective, it suffices to prove that any f is homotopic to some w_k.

Let $f \in [f] \in [S^1, S^1]$ be an arbitrary map, regarded as a map from $[0,1]$ to S^1 such that $f(0) = f(1) = 1 \in S^1$. Then the lift $\widetilde{f}$ is a (continuous) function defined on $[0,1]$ with values in $\mathbb{R}$. Let $\deg(f) = k$, i.e., $\widetilde{f}(1) = 2k\pi$. The graph of $\widetilde{f}$ is shown on Fig. 1(b). The graph of w_k is a straight line joining the points $(0,0)$ and $(1, 2k\pi)$. There is an obvious homotopy (shown in the figure) joining w_k and $\widetilde{f}$; denote it by $G(x,t)$. Then $F(x,t) := \exp(G(x,t))$ is the required homotopy between f and w_k. $\square$

Corollary. *The identity map of the circle is not homotopic to the constant map* $S^1 \to 1 \in S^1$.

§4. Connections with other topological problems

The theorem proved in the previous section allows to give a rigorous definition of the index of a vector field (see Lecture 6) as the degree of the Gauss map $S^1 \to S^1$. (The definition in Lecture 6 is not rigorous, it was given in intuitive terms.)

What is more important, the Corollary to the theorem is essentially equivalent to several other basic topological statements, namely the Brouwer fixed-point theorem, the "drum lemma", the nonexistence of vector fields without singular points on the sphere, etc. For details see the exercise assignments.