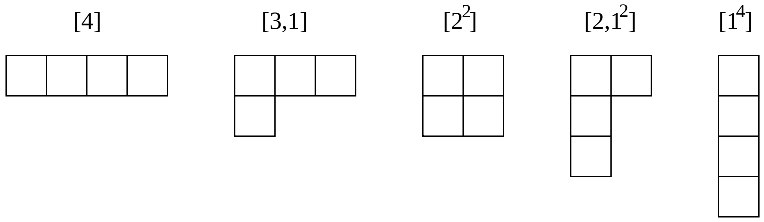
S. V. Duzhin

COMBINATORICS OF SYMMETRIC GROUPS

Lecture 1: Partitions

A partition of a non-negative integer n is a nonincreasing sequence of positive integers $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. In case of repeating elements, exponential notation is used: $[2,1^2]=[2,1,1]$. For example, there are 5 partitions of 4; we list them below together with their graphical representation by means of *Young diagrams*:



The Young diagram of a partition $[\lambda_1, \lambda_2, \ldots, \lambda_k]$ is made up of λ_1 cells in the first row, λ_2 cells in the second row, etc., all left justified. It is merely a graphical means to visualize the partition.

Let $p(n)$ be the number of partitions of n. Here is the table of values of $p(n)$ for small n:

n	0	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	1	2	3	5	7	11	15	22	30	42

(The value $p(0) = 1$ has the following meaning: *there is a unique partition of 0 into positive integers, viz. the partition into an empty set of parts.*)

The generating function of the sequence $p(n)$ can be written as an infinite product invented by L. Euler:

Theorem 1.

$$\sum_{n_0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} \tag{1}$$

Exercise 1. Prove this formula.

Hint. Expand each factor $1/(1-x^k)$ as a series and think about the coefficient of x^n in the infinite product.

Exercise 2. Prove that the number of partitions of a given number n into odd parts is equal to the number of partitions of n into different parts. (For example, $\#\{[5],[3,1^2],[1^5]\} = \#\{[5],[4,1],[3,2]\}$.)

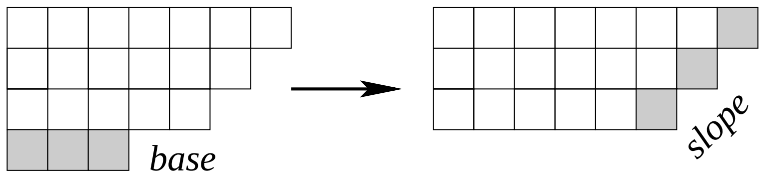
Hint. Modify appropriately the proof of Theorem 1.

The denominator in the right hand side of equation 1 expands as follows.

Theorem 2. (*Euler's pentagonal theorem.*)

$$\prod_{k=1}^{\infty}(1-x^k)=\sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k+1)}{2}}=1+\sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{k(3k-1)}{2}}+x^{\frac{k(3k+1)}{2}}\right). \tag{2}$$

Idea of the proof. The coefficient of x^n in the expansion of the left side product is equal to $\#E_n-\#O_n$, where E_n is the set of partitions of n into an even number of different summands, while O_n is the set of partitions of n into an odd number of different summands. The crucial observation is that for almost all n (in fact, for all n except the "pentagonal" numbers $n=k(3k+1)/2$, $k\in\mathbb{Z}$) there is a one-to-one correspondence between E_n and O_n established by either cutting off the "base" of the Young diagram and attaching it to its "slope" or, vice versa, cutting off the "slope" and attaching it to the "base":



Corollary (recurrent formula for the number of partitions). For any $n>0$ we have:

$$\begin{aligned} p(n) &= \sum_{k=1}^{\infty}(-1)^k\Big[p\big(n-\frac{k(3k-1)}{2}\big)+p\big(n-\frac{k(3k+1)}{2}\big)\Big] \\ &= p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\ldots, \end{aligned}$$

where $p(n)=0$, if $n<0$.

Exercise 3. Let $\sigma(n)$ denote the sum of all divisors of n. Find a relation between $\sigma(n)$ and $p(n)$ and deduce a recurrent formula for $\sigma(n)$, similar to the previous Corollary.

Euler's identity 2 inspired a lot of investigations on "combinatorial identities" that led to a host of beautiful formulas. For example:

Jacobi's triple identity.

$$\prod_{k=1}^{\infty}(1-x^k y^k)(1-x^{k-1}y^k)(1-x^k y^{k-1})=\sum_{n=-\infty}^{\infty}(-1)^n x^{n(n+1)/2}y^{n(n-1)/2}. \tag{3}$$

Exercises. (a) Prove Jacobi's identity using Eq. (2). (b) Vice versa, deduce the pentagonal theorem as a special case of Jacobi's identity. (c) Differentiate (3) over y at $y=1$ and write the result as a formula for $\prod_{k=1}^{\infty}(1-x^k)^3$.

Definition. A *plane partition* of the number n is a set of positive integers with total sum n, arranged as a Young diagram with nonincreasing rows and columns.

Example. $\begin{array}{|c|c|c|c|c|} \hline 4 & 4 & 3 & 1 & 1 \\ \hline 2 & 2 & 1 \\ \cline{1-3} 2 & 1 \\ \cline{1-2} \end{array}$ is a plane partition of 21.

Denote by $\pi(n)$ the number of plane partitions of n. For example, $p(0) = p(1) = 1$, $p(2) = 3$, $p(3) = 6$.

Theorem 3. (MacMahon).

$$\sum_{n=0}^{\infty} \pi(n) x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k}.$$