Diagrams.

All the problems below are about Abelian groups and/or modules over a commutative ring k

1◇1. Which of the following statements about an commutative triangle
$$\begin{tikzcd} U \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & W \\ & V \arrow[ur, "\beta"'] & \end{tikzcd}$$
are true?

a) α, β are epi $\Rightarrow \gamma$ is epi **b)** α, β are mono $\Rightarrow \gamma$ is mono **c)** γ is epi $\Rightarrow \alpha$ is epi **d)** γ is epi $\Rightarrow \beta$ is epi
e) γ is mono $\Rightarrow \alpha$ is mono **f)** γ is mono $\Rightarrow \beta$ is mono **g)** if α is epi, then γ is epi $\iff \beta$ is epi
h) if α is epi, then γ is mono $\iff \beta$ is mono **i)** if β is epi, then γ is epi $\iff \alpha$ is epi
j) if β is epi, then γ is mono $\iff \alpha$ is mono **k)** if γ is an isomorphism, then α is mono and β is epi.

1◇2. Consider a commutative diagram with exact rows:
$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] \arrow[d, "\varphi'"] & V \arrow[r] \arrow[d, "\varphi"] & V'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & W' \arrow[r] & W \arrow[r] & W'' \arrow[r] & 0 \end{tikzcd} \tag{1}$$
Which of the implications below are true?
a) φ is an isomorphism $\iff \varphi'$ is mono and φ'' is epi
b) φ', φ'' are isomorphisms $\Rightarrow \varphi$ is an isomorphism

1◇3. Show that there is a unique morphism $V' \xrightarrow{\varphi'} W'$ fitted commutatively in the following commutative diagram with exact rows:
$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] & V \arrow[r] \arrow[d, "\varphi"] & V'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & W' \arrow[r] & W \arrow[r] & W'' \arrow[r] & 0 \end{tikzcd}$$
Find sufficient conditions implying injectivity of φ'?

1◇4. Do the same for the diagram
$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] \arrow[d, "\varphi'"] & V \arrow[r] \arrow[d, "\varphi"] & V'' \arrow[r] & 0 \\ 0 \arrow[r] & W' \arrow[r] & W \arrow[r] & W'' \arrow[r] & 0 \end{tikzcd}$$
but find sufficient conditions for surjectivity of φ''.

1◇5 (5 morphisms lemma). Consider commutative diagram with exact rows
$$\begin{tikzcd} V_1' \arrow[r] \arrow[d, "\varphi_1'"'] & V_2' \arrow[r] \arrow[d, "\varphi_2'"'] & V \arrow[r] \arrow[d, "\varphi"] & V_2'' \arrow[r] \arrow[d, "\varphi_2''"] & V_1'' \arrow[d, "\varphi_1''"] \\ W_1' \arrow[r] & W_2' \arrow[r] & W \arrow[r] & W_2'' \arrow[r] & W_1'' \end{tikzcd}$$
Does bijectivity (accord. injectivity, surjectivity) of 4 flank arrows φ_1', φ_1'', φ_2', φ_2'' imply the same property of the middle arrow φ?

1◇6 (snake lemma). Show that the commutative diagram (1) (with exact rows) induces a long exact sequence:
$0 \longrightarrow \ker\varphi' \longrightarrow \ker\varphi \longrightarrow \ker\varphi'' \longrightarrow \operatorname{coker}\varphi' \longrightarrow \operatorname{coker}\varphi \longrightarrow \operatorname{coker}\varphi'' \longrightarrow 0.$

1◇7 (long (co)homology sequence). Show that each short exact sequence of complexes
$$0 \longrightarrow K'_\bullet \xrightarrow{\alpha'} K_\bullet \xrightarrow{\alpha''} K''_\bullet \longrightarrow 0$$
(α's commute with the differentials) induces a long exact sequence of their homologies
$$\cdots \longrightarrow H_i(K'_\bullet) \xrightarrow{\alpha'_*} H_i(K_\bullet) \xrightarrow{\alpha''_*} H_i(K''_\bullet) \xrightarrow{\delta} H_{i-1}(K'_\bullet) \xrightarrow{\alpha'_*} H_{i-1}(K_\bullet) \longrightarrow \cdots$$

Projective resolutions

(all the problems below are about Abelian groups and/or modules over a commutative ring k)

2◇1 (homotopies). Let $K_\bullet \xrightarrow{f} L_\bullet$ and $K_\bullet \xrightarrow{g} L_\bullet$ be two morphisms of complexes. By the definition, a *homotopy* between f and g is a sequence of maps $K_i \xrightarrow{\varphi_i} L_{i+1}$ such that $f - g = \partial\circ\varphi + \varphi\circ d$ (see (2)).

$$\begin{tikzcd} \cdots \arrow[r, "d"] & K_{i+1} \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & K_i \arrow[r, "d"] \arrow[dl, squiggly, "\varphi_i"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & K_{i-1} \arrow[r, "d"] \arrow[dl, squiggly, "\varphi_{i-1}"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \cdots \\ \cdots \arrow[r, "\partial"] & L_{i+1} \arrow[r, "\partial"] & L_i \arrow[r, "\partial"] & L_{i-1} \arrow[r, "\partial"] & \cdots \end{tikzcd} \tag{2}$$

Show that if such a homotopy exists, then the maps $H_\bullet(K_\bullet) \xrightarrow{\overline{f}} H_\bullet(L_\bullet)$ and $H_\bullet(K_\bullet) \xrightarrow{\overline{g}} H_\bullet(L_\bullet)$, which are induced by f and g on the homologies, coincide.

2◇2. For any two complexes, consider modules $Map_i(K_\bullet, L_\bullet) \stackrel{\text{def}}{=} \bigoplus_\nu \operatorname{Hom}(K_\nu, L_{\nu-i})$ (here are *all* the maps, not necessary commuting with differentials). Consider the maps $Map_i(K_\bullet, L_\bullet) \xrightarrow{\delta} Map_{i-1}(K_\bullet, L_\bullet)$ sending $\varphi \longmapsto (\partial\circ\varphi - (-1)^i\varphi\circ d)$. Show that **a)** $Map_i(K_\bullet, L_\bullet)$ form a complex w.r.t. δ
b) $H_0(Map_\bullet(K_\bullet, L_\bullet))$ consists of all morphisms of complexes $K_\bullet \xrightarrow{f} L_\bullet$ considered up to homotopy.

2◇3. Show that for any exact triple $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ and any P there exists an exact sequence

$$0 \to \operatorname{Hom}(P, A) \xrightarrow{\alpha_*} \operatorname{Hom}(P, B) \xrightarrow{\beta_*} \operatorname{Hom}(P, C) \tag{3}$$

where $\alpha_* : \varphi \longmapsto \alpha\circ\varphi$ and $\beta_* : \varphi \longmapsto \beta\circ\varphi$. Give an example showing that β_* is non surjective in general.

<u>Definition.</u> Module P is called *projective*, if for any epimorphism $B \overset{\beta}{\twoheadrightarrow} C$ and any morphism $P \xrightarrow{\psi} C$ there exists a morphism $P \xrightarrow{\varphi} B$ such that $\beta\circ\varphi = \psi$, that is, if β_* in (3) is surjective.

2◇4. Let P be a projective factor module of an arbitrary module M. Show that $M = P \oplus M'$.

2◇5. Show that P is projective iff there exists M such that $P \oplus M$ is free. Show that free modules are projective.

2◇6 (existence of a projective resolution). Show that any M can be included in an exact sequence (maybe non unique and unbounded from the left side) $\cdots \xrightarrow{d} F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{d} M \to 0$, where all F_ν are free. The complex: $\cdots \xrightarrow{d} F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \to 0$ is called *a free resolution* for M (it is obtained by removing M and is exact everywhere except for F_0 where its homology equals M).

2◇7 (functoriality of a projective resolution). Let $P_\bullet$ and $Q_\bullet$ be projective resolutions for modules M and N respectively. Show that for any morphism $M \xrightarrow{\overline{\varphi}} N$ there exists a unique up to a homotopy morphism of complexes $P_\bullet \xrightarrow{f} Q_\bullet$ that induces $\overline{\varphi}$ on the homology.

2◇8 (functors $\mathbf{Ext}^i(M, N)$). Let $P_\bullet$ be a projective resolution for module M. Show that for any N the modules $K^i = \operatorname{Hom}(P_i, N)$ form a complex whose cohomologies do not depend on the choice of a resolution and are functorial in M. They are denoted by $Ext^i(M, N)$.

2◇9. Show that elements of $Ext^1(M, N)$ are in 1–1 correspondence with exact triples

$$0 \to N \to X \to M \to 0$$

considered up to isomorphisms identical on M and N.

2◇10. Let $K = k[x, y]$ be the polynomial ring over a field k. Consider 3 K-modules: $K = k[x, y]$, $L = k[x, y]/(x)$, $M = k[x, y]/(x, y)$. Compute all Ext's between them.

Koszul resolutions

3⋄1. Construct a free resolution[1] for $k[x_1,x_2,x_3]$-module $k[x_1,x_2,x_3]/(x_1,x_2,x_3)$.

3⋄2* (twisted cubic). Show that a cubic polynomial $a_0t_0^3+a_1t_0^2t_1+a_2t_0t_1^2+a_3t_1^3$ equals a cube of some linear form $\alpha_0t_0+\alpha_1t_1$ iff

$$\operatorname{rk}\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix} = 1\,. \tag{4}$$

Try to extend this result onto homogeneous binary forms $f(t_0,t_2)$ of degree $d>2$.

3⋄3. Construct a free resolution for the ideal in $k[a_0,a_1,a_2,a_3]$ spanned by the quadratic equations (4) from the previous problem.

3⋄4 (partially dualized Koszul complex). Let V be n-dimensional vector space over a field K of zero characteristic. Consider differential forms on V with polynomial coefficients. If we write $x_1,x_2,\dots,x_n$ for coordinates in V and $\xi_\nu=dx_\nu$ for the corresponding differentials, then homogeneous degree m differential k-form looks like $\sum\limits_{i_1<i_2<\cdots<i_k} f(x_1,x_2,\dots,x_n)\,\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\wedge\xi_{i_k}$ and these forms form a vector space denoted by $S^mV^*\otimes\Lambda^kV^*$. Consider the map

$$\partial \stackrel{\text{def}}{=} \sum x_i\frac{\partial}{\partial\xi_i} : S^mV^*\otimes\Lambda^kV^* \to S^{m+1}V^*\otimes\Lambda^{k-1}V^* \tag{5}$$

taking $f(x_1,x_2,\dots,x_n)\,\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\wedge\xi_{i_k}$ to $\sum\limits_{\nu=1}^{k}(-1)^{\nu-1}x_{i_k}f(x_1,x_2,\dots,x_n)\,\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\wedge\widehat{\xi_{i_\nu}}\wedge\cdots\wedge\xi_{i_k}$,

where $\widehat{\xi_{i_\nu}}$ means that this multiplier should be removed from the product.

a) Show that after summation over m we get (identifying $\oplus S^mV^*$ with $K[x_1,x_2,\dots,x_n]$) a complex of free $K[x_1,x_2,\dots,x_n]$-modules

$$\begin{gathered}0\to\Lambda^nV^*\otimes K[x_1,x_2,\dots,x_n]\to\Lambda^{n-1}V^*\otimes K[x_1,x_2,\dots,x_n]\to\Lambda^2V^*\otimes K[x_1,x_2,\dots,x_n]\to\ \cdots\\ \cdots\ \to\Lambda^1V^*\otimes K[x_1,x_2,\dots,x_n]\to\Lambda^0V^*\otimes K[x_1,x_2,\dots,x_n]\to 0\end{gathered} \tag{6}$$

b) De Rahm maps

$$d \stackrel{\text{def}}{=} \sum \xi_i\frac{\partial}{\partial x_i} : S^mV^*\otimes\Lambda^kV^* \to S^{m-1}V^*\otimes\Lambda^{k+1}V^* \tag{7}$$

take $f(x_1,x_2,\dots,x_n)\,\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\wedge\xi_{i_k}$ to $\sum\limits_{i=1}^{n}\dfrac{\partial f}{\partial x_i}(x_1,x_2,\dots,x_n)\,\xi_i\wedge\xi_{i_1}\wedge\xi_{i_2}\wedge\cdots\wedge\xi_{i_k}$ After summation over m, they form a sequence of degree -1 arrows from the complex (6) to itself. Compute the supercommutators $\partial d+d\partial$ and show that d's give a homotopy between some natural K-linear automorphism of (6) and some map which equals zero on all $\Lambda^kV^*\otimes K[x_1,x_2,\dots,x_n]$ with $k\geqslant 1$.

c) Compute homologies of the complex (6).

d) Compare the map (5) with the composition

$$S^mV^*\otimes\Lambda^kV^* \xrightarrow{c} S^mV^*\otimes V^*\otimes\Lambda^{k-1}V^* \xrightarrow{\mu} S^{m+1}V^*\otimes\Lambda^{k-1}V^*$$

where $S^mV^*\otimes V^*\xrightarrow{\mu}S^{m+1}V^*$ is multiplication of polynomials and $\Lambda^kV^*\xrightarrow{c}V^*\otimes\Lambda^{k-1}V^*$ is given by the same element

$$c\in\operatorname{Hom}_K(\Lambda^kV^*,V^*\otimes\Lambda^{k-1}V^*)=(\Lambda^kV^*)^*\otimes V^*\otimes\Lambda^{k-1}V^*=\operatorname{Hom}_K(\Lambda^kV^*\otimes V,\Lambda^{k-1}V^*)$$

[1] i. e. find generators, basic relations, basic relations between relations and so on

as the canonical map $\Lambda^k V^* \otimes V \xrightarrow{c} \Lambda^{k-1}V^*$ called «the contraction in the first index».

3◇5 (Koszul complex). Let V be n-dimensional vector space over a field K of any characteristic $\mathit{char} K \neq 2$ and let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{x_1, x_2, \dots, x_n\} \subset V^*$ be a pair of dual bases. Then the identity map $Id_V \in \mathrm{Hom}_K(V,V) = V^* \otimes V$ can be written as $\sum x_i \otimes e_i$ and multiplication by this element gives a map

$$\partial : S^m V^* \otimes \Lambda^k V \to S^{m+1}V^* \otimes \Lambda^{k+1}V \tag{8}$$

which takes $f(x_1, x_2, \dots, x_n)\, e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ to $\sum_{i=1}^{k} x_i f(x_1, x_2, \dots, x_n)\, e_i \wedge e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ Show that taking a sum over m we get a complex of free $K[x_1, x_2, \dots, x_n]$-modules

$$0 \to \Lambda^0 V \otimes K[x_1, x_2, \dots, x_n] \to \Lambda^1 V \otimes K[x_1, x_2, \dots, x_n] \to \Lambda^2 V \otimes K[x_1, x_2, \dots, x_n] \to \cdots$$
$$\cdots \to \Lambda^{n-1} V \otimes K[x_1, x_2, \dots, x_n] \to \Lambda^n V \otimes K[x_1, x_2, \dots, x_n] \to 0 \tag{9}$$

and compute its homologies using appropriate De Rahm like homotopy.

3◇6*. Let $\mathit{char} K = 0$, fix a base element $\delta \in \Lambda^n V$ and identify $\Lambda^k V \simeq (\Lambda^{n-k}V)^* \simeq \Lambda^{n-k}V^*$ using exterior multiplication and full polarization. Compare Koszul complexes constructed in two previous problems.

3◇7 (Koszul complex of an ideal). Let K be any commutative ring and $I \subset K$ be an ideal spanned by some $f_1, f_2, \dots, f_n \in K$. Let $F = K^{\oplus n}$ be a free K-module of rank n with a fixed base $\{e_1, e_2, \dots, e_n\}$. Show that the maps $\Lambda^k F \otimes K \to \Lambda^{k+1} F \otimes K$ sending $g \otimes e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ to $\sum_{i=1}^{n} f_i g \otimes e_i \wedge e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ produce a complex of free K-modules

$$0 \to \Lambda^0 F \otimes K \to \Lambda^1 F \otimes K \to \Lambda^2 F \otimes K \to \cdots \to \Lambda^{n-1} F \otimes K \to \Lambda^n F \otimes K \to 0 \tag{10}$$

which coincides with the tensor product of n short complexes $0 \to K \xrightarrow{x \mapsto f_i x} K \to 0$ whose two terms are situated at degrees 0 and 1.

3◇8. Let $f \in K$ and $L^\bullet_f$ be a tensor product of an arbitrary complex $L^\bullet$ with the short complex $0 \to K \xrightarrow{x \mapsto fx} K \to 0$ whose two terms are situated at degrees 0 and 1. Construct a long exact sequence

$$\cdots \to H^{i-1}(L^\bullet) \xrightarrow{x \mapsto fx} H^{i-1}(L^\bullet) \to H^i(L^\bullet_f) \to H^i(L^\bullet) \xrightarrow{x \mapsto fx} H^i(L^\bullet) \to \cdots$$

3◇9* (regular sequences). A sequence $(f_1, f_2, \dots, f_n) \subset K$ (its ordering is important) is called *regular*, if for each $i = 2, 3, \dots, n$ the class $f_i \bmod (f_1, f_2, \dots, f_{i-1})$ is not a zero divisor in $K/(f_1, f_2, \dots, f_{i-1})$.

a) Show that if $(f_1, f_2, \dots, f_n)$ is regular, then its Koszul complex (10) is exact.

b) Let $K = k[x,y,z]$, $f_1 = x$, $f_2 = y(1-x)$, $f_3 = z(1-x)$. Clear up, are the sequences (f_1, f_2, f_3) and (f_2, f_3, f_1) regular or not?

c) Let $K = k[x,y,z]$, $f_1 = x(1-2xyz)(1-3xyz)$, $f_2 = y(1-xyz)(1-3xyz)$, $f_3 = z(1-xyz)(1-2xyz)$. Show that the corresponding Koszul complex is exact but all permutations of (f_1, f_2, f_3) are not regular.

d) Let K be a local ring[2], $f_1, f_2, \dots, f_n$ belong to its maximal ideal, and the Koszul complex (10) be exact. Show that $(f_1, f_2, \dots, f_n)$ is regular.

3◇10*. Let $I = (f_1, f_2, \dots, f_n) \subset K$ coincide with the whole of K. Show that (10) is exact.

3◇11 (Koszul complex of a module). Let K be any commutative ring, M be a finitely generated K-module, and $M \xrightarrow{\varphi} K$ be K-linear homomorphism. Show that the maps

$$\partial_\varphi \;:\; \Lambda^k M \to \Lambda^{k-1} M \tag{11}$$

taking $m_1 \wedge m_2 \wedge \dots \wedge m_k$ to $\sum_{\nu=1}^{k} (-1)^{\nu-1} \varphi(m_\nu)\, m_1 \wedge m_2 \wedge \dots \wedge \widehat{m}_\nu \wedge \dots \wedge m_k$ from a complex whose zero homology (at the rightmost term) equals $K/\mathrm{im}\,\varphi$. This complex is called *a Koszul complex* of $M \xrightarrow{\varphi} K$ and their homologies are denoted by $H_\bullet(\varphi)$.

3◇12. Under the previous problem conditions, let $N = M \oplus K$ and $N \xrightarrow{\psi} K$ send (m, f) to $\varphi(m) + af$ for some fixed $a \in K$. Construct a long exact sequence of the Koszul cohomologies

$$\cdots \to H_{k+1}(\psi) \to H_k(\varphi) \xrightarrow{m \mapsto am} H_k(\varphi) \to H_k(\psi) \to H_{k-1}(\varphi) \to \cdots$$

[2] i. e. it has a unique maximal ideal and has not zero divisors

Spectral sequences

4◇1 (coupled pair). An exact triangle of module homomorphisms
$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$
is called *a coupled pair* and denoted by (D, E, i, j, k). Show that $d = jk : E \to E$ satisfies $d^2 = 0$ and a triangle
$$\begin{tikzcd} D' \arrow[rr, "i"] & & D' \arrow[dl, "j"] \\ & E' \arrow[ul, "k"] & \end{tikzcd}$$
with $D' = \operatorname{im} i \subset D$, $E' = \ker d/\operatorname{im} d$, $i' = i|_{\operatorname{im} i}$, $j' : i(x) \longmapsto j(x)$, $k' : x \pmod{\operatorname{im} d} \longmapsto k(x)$ is well defined and exact, i.e. (D', E', i', j', k') is a coupled pair as well (it is called *the derived pair* of (D, E, i, j, k)).

4◇2. Let $(D_r, E_r, i_r, j_r, k_r)$ be $(r-1)$-th successive derivative of a coupled pair $(D_1, E_1, i_1, j_1, k_1)$. Show that $D_r = \operatorname{im} i_1^{r-1}$, $E_r = k_1^{-1}(\operatorname{im} i_1^{r-1})/j_1(\ker i_1^r)$ and there is an exact sequence
$$0 \to \operatorname{im} i_1^{r-1}/\operatorname{im} i_1^r \to E_r \to \ker i_1^r/\ker i_1^{r-1} \to 0$$

<u>Bigraded coupled pairs & spectral sequences.</u> A coupled pair (D, E, i, j, k) is called *upper-style bigraded*, if $D = \bigoplus\limits_{p,q\in\mathbb{Z}} D^{p,q}$, $E = \bigoplus\limits_{p,q\in\mathbb{Z}} E^{p,q}$ and $\deg i = (-1, 1)$, $\deg i = (0, 0)$, $\deg i = (1, 0)$, that is
$$\begin{aligned} i &= \oplus i^{(p,q)}\,, \text{ where } i^{(p,q)} : D^{p,q} \to D^{p-1,q+1} \\ j &= \oplus j^{(p,q)}\,, \text{ where } j^{(p,q)} : D^{p,q} \to E^{p,q} \\ k &= \oplus k^{(p,q)}\,, \text{ where } k^{(p,q)} : E^{p,q} \to D^{p+1,q} \end{aligned}$$
Similarly, a *bottom-style bigraded* coupled pair (D, E, i, j, k) has $D = \bigoplus\limits_{p,q\in\mathbb{Z}} D_{p,q}$, $E = \bigoplus\limits_{p,q\in\mathbb{Z}} E_{p,q}$ and homogeneous morphisms of degrees $\deg i = (1, -1)$, $\deg j = (0, 0)$, $\deg k = (-1, 0)$. These two grading styles go to each other by changing signs of the both indices. We will use upper-style notations in what follows. Let $(D_r, E_r, i_r, j_r, k_r)$ with $r = 1, 2, \dots$ be a sequence of successive derivatives for an initial bigraded coupled pair $(D, E, i, j, k) = (D_1, E_1, i_1, j_1, k_1)$. For each r, we write direct summands $E_r^{p,q}$, of E_r, into the cells of an infinite rectangular table and number these cells by the pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ (taking p as the horizontal coordinate). The sequence of these tables $E_0, E_1, E_2, \dots$ is called *a spectral sequence* of bigraded coupled pair (D, E, i, j, k).

4◇3. Show that r-th table E_r is equipped with the differentials $d_r^{(p,q)} = (j_r \circ k_r)^{(p,q)} : E_r^{p,q} \to E_r^{p+r,q+1-r}$ and $(r+1)$-th table is obtained from r-th one by taking the cohomology $E_{r+1}^{p,q} = \ker d_r^{(p,q)}/\operatorname{im} d_r^{p-r,q-1+r}$ at each cell (see the diagram (12) below).

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, "k"] & D^{p-1,q+2} \arrow[r, "j"] & E^{p-1,q+2} \arrow[r, "k"] & D^{p,q+2} \arrow[r, "j"] & E^{p,q+2} \arrow[r, "k"] & D^{p+1,q+2} \arrow[r, "j"] & E^{p+1,q+2} \arrow[r, "k"] & D^{p+2,q+2} \arrow[r, "j"] & {} \\
{} \arrow[r, "k"] & D^{p-1,q+1} \arrow[r, "j"] & E^{p-1,q+1} \arrow[r, "k"] & D^{p,q+1} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & E^{p+1,q+1} \arrow[r, "k"] & D^{p+2,q+1} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & {} \\
{} \arrow[r, "k"] & D^{p-1,q} \arrow[r, "j"] & E^{p-1,q} \arrow[r, "k"] & D^{p,q} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & E^{p,q} \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & {} \\
{} \arrow[r, "k"] & D^{p-1,q-1} \arrow[r, "j"] & E^{p-1,q-1} \arrow[r, "k"] & D^{p,q-1} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & E^{p,q-1} \arrow[r, "k"] & D^{p+1,q-1} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & E^{p+1,q-1} \arrow[r, "k"] & D^{p+2,q-1} \arrow[r, "j"] \arrow[ull, bend left, "i"'] & {}
\end{tikzcd} \tag{12}$$

4◇4. Let $E_1^{p,q}$ vanish for all $q \ll 0$ (uniformly in p) and for all $p \ll 0$ (uniformly in q).

a) What does this mean in terms of $D^{p,q}$ with fixed $(p+q) = n$ and $p \ll 0$ and $p \gg 0$?

b) Show that $\forall\, n\ \exists\, N$ such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r > N$ and all p, q with $p + q = n$ (these stabilized terms are called *limit terms* and denoted by $E_\infty^{p,q}$).

c) Describe $E_\infty^{p,q}$ and $\bigoplus\limits_{p+q=n} E_\infty^{p,q}$ in terms of the map $D \xrightarrow{i} D$.

Limits We say that a spectral sequence $E_r^{p,q}$ converges to E_∞^n and write $E_r^{p,q} \Rightarrow E_\infty^n$, if two conditions hold:

1. $E_r^{p,q}$ becomes stabilize, i.e. $\forall\ p,q\ \exists\ N = N(p,q)$ such that both outgoing and incoming differentials at (p,q)-cell vanish in all terms E_r with $r > N$; then the stable entry $E_{N+1}^{p,q} = E_{N+2}^{p,q} = \dots$ is denoted by $E_\infty^{p,q}$
2. E_∞^n has a decreasing filtration $F^pE_\infty^n$ whose factors coincide with stable $E_\infty^{p,n-p}$, i.e. $E_\infty^n = \bigcup\limits_p F^pE_\infty^n$, where $\cdots \xrightarrow{\supset} F^{p-1}E_\infty^n \supset F^pE_\infty^n \supset F^{p+1}E_\infty^n \supset \cdots$ and $F^pE_\infty^n/F^{p+1}E_\infty^n = E_\infty^{p,n-p}$

4◇5 (filtered complexes). Let each term K^m, of a complex $\cdots \xrightarrow{d} K^m \xrightarrow{d} K^{m+1} \xrightarrow{d} \cdots$, have a decreasing filtration $K^m = F^0K^m \supset F^1K^m \supset F^2K^m \subset \cdots$ such that $d(F^pK^m) \subset F^pK^{m+1}$ for all p, m, that is, for each p there is a complex $F^pK^\bullet$ with the same differential d.

a) Show that for each p there is a factor complex $G^pK^\bullet$ whose m'th term equals $F^pK^m/F^{p+1}K^m$ and the differential is induced by d.

b) For each p, construct a long exact sequence of cohomologies

$$\cdots\ H^i(F^{p+1}K^\bullet) \to H^i(F^pK^\bullet) \to H^i(G^pK^\bullet) \to H^{i+1}(F^{p+1}K^\bullet) \to \cdots$$

c) Construct a bigraded coupled pair with $D_1^{p,q} = H^{p+q}(F^pK^\bullet)$, $E_1^{p,q} = H^{p+q}(G^pK^\bullet)$ and describe $E_r^{p,q}$-terms of all its successive derivatives.

d) Show that if the complex is bounded and all the filtrations are uniformly bounded, then there exists a spectral sequence $E_r^{p,q} \Rightarrow H^n(K^\bullet)$ with $E_1^{p,q} = H^{p+q}(G^pK^\bullet)$.

4◇6 (double complexes). A *a double complex* is a bigraded object $K^{\bullet\bullet} \oplus K^{p,q}$ equipped with two differentials $K^{p,q} \xrightarrow{d_1} K^{p+1,q}$, $K^{p,q} \xrightarrow{d_2} K^{p,q+1}$ that satisfy $d_1^2 = d_2^2 = 0$ and supercommute, i.e. $d_1d_2 + d_2d_1 = 0$. Show that:

a) objects $Tot^n(K) \stackrel{\text{def}}{=} \bigoplus\limits_{p+q=n} K^{p,q}$ equipped with the maps $Tot^n(K) \xrightarrow{d_1+d_2} Tot^{n+1}(K)$ form a complex (it is called *a total complex* or *a contraction* of the double complex $K^{\bullet\bullet}$)

b) the complex $Tot^\bullet(K)$ has two decreasing filtrations:

$${}^{I}\!F^p\, Tot^m(K) = \bigoplus_{\substack{\nu\geqslant p\\ p+q=n}} K^{p,q} \qquad \text{and} \qquad {}^{II}\!F^q\, Tot^m(K) = \bigoplus_{\substack{\nu\geqslant q\\ p+q=n}} K^{p,q}$$

c) p-th factor complex ${}^{I}\!G^p\,Tot^\bullet(K) = \bigoplus\limits_{\substack{\nu\geqslant p\\ p+q=n}} K^{p,q}$, of the first filtration, coincides with the p-th column of the bicomplex $K^{\bullet\bullet}$ considered as a complex w.r.t. the differential d_2 (we'll write $H_{d_2}^q(K^{p,\bullet})$ for its q-th cohomology)

d) the differentials d_1 induce well defined morphisms $H_{d_2}^q(K^{p,\bullet}) \xrightarrow{d_1} H_{d_2}^q(K^{p+1,\bullet})$ between these cohomologies, that is, produce a complex of them (we'll write $H_{d_1}^p(H_{d_2}^q(K))$ for its p-th cohomology)

e) prb. 4◇5 leads to a spectral sequence ${}^{I}\!E_r^{p,q} \Rightarrow H^n(Tot^\bullet(K))$ with ${}^{I}\!E_2^{p,q} = H_{d_1}^p(H_{d_2}^q(K))$

f) there is the similar spectral sequence ${}^{II}\!E_r^{p,q} \Rightarrow H^n(Tot^\bullet(K))$ with ${}^{II}\!E_2^{p,q} = H_{d_2}^q(H_{d_1}^p(K))$ coming from the second filtration on $Tot^\bullet(K)$

4◇7. Produce the both spectral sequences of a double complex from appropriate coupled pairs without any usage of prb. 4◇5

4◇8 (Cartan - Eilenberg resolution). Let $K_\bullet$ be a complex of modules. Construct a double complex $F_{\bullet\bullet}$ such that each column complex $F_{\bullet,p}$ (w.r.t. the differntial d_2) has only one nonzero homology $H_0(F_{\bullet,p}) = K_\bullet$ and the row complex formed by these homologies (w.r.t. the differential d_1) coincides with $K_\bullet$. The total complex $Tot_\bullet(F)$ is called a (free) *Cartan - Eilenberg resolution* for the complex $K_\bullet$.

4◇9. Show that each morphism of complexes can be extended to a morphism of their Cartan - Eilenberg resolutions and this extension is unique up to a homotopy.

4◇10. Let a complex of modules $K_\bullet$ have only one non zero homology $H_0(K_\bullet) = M$ and let a module N be such that $Ext^\nu(K_i, N) = 0$ for all i and all $\nu \geqslant 1$. Using a Cartan - Eilenberg resolution for $K_\bullet$, show that $Ext^m(M,N)$ coincides with m-th cohomology group of the complex $\mathrm{Hom}(K_\bullet, N)$.