Geometry of symmetric tensors[1]

Polynomial functions and symmetric multilinear forms. Recall that each $f \in S^nV^*$ produces:

- (anyway) k-valued polynomial function $f(v)$, in one variable $v \in V$
- (as soon $char(k) = 0$) k-valued symmetric multilinear form $\widetilde{f}(v_1, v_2, \dots, v_n) = \langle\, \mathrm{pl}(f)\,,\, v_1 \otimes v_2 \otimes \cdots \otimes v_n \,\rangle$, in n variables $v_i \in V$, where $\mathrm{pl}(f) \in \mathsf{ST}^nV^* \subset V^{*n}$ is the full polarization of f (see 'Tensor Guide').

They are uniquely defined by each other via the prescription: $f(v) = \widetilde{f}(v, v, \dots, v)$. If $f = \xi_1\xi_2 \dots \xi_n$ is a product of linear forms, then $\widetilde{f}(v) = f(v, v, \dots, v) = \langle\, \mathrm{sym}_n\,(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n)\,,\, v^{\otimes n} \,\rangle = \prod_\nu \xi_\nu(v)$.

Polarities and contractions. For any vector $v \in V$ and any polynomial $f \in S^nV^*$ the contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ does not depend on the choice of index i in $\mathrm{pl}(f)$ to be contracted, because $\mathrm{pl}(f) \in \mathsf{ST}^nV^*$ is symmetric. In terms of multilinear forms, $c_1^i(\mathrm{pl}(f) \otimes v) = i_v(\widetilde{f})$ is the inner product of $\widetilde{f}$ and v (as in the Example 4.9), i. e. symmetric multilinear form in $(n-1)$ arguments on V that takes $(w_1, w_2, \dots, w_{n-1})$ to $\widetilde{f}(v\,, w_1, w_2, \dots, w_{n-1}\,)$. A projection of $c_1^i(\mathrm{pl}(f) \otimes v)$ from $\mathsf{ST}^{n-1}V^*$ to $S^{n-1}V^*$ is called *a polar* of v w. r. t. f and is denoted by $\mathrm{pl}_v f$. This is degree $(n-1)$ homogeneous polynomial in one variable $w \in V$:

$$\mathrm{pl}_v f(w) = \widetilde{f}(v, w, w, \dots, w)\ .$$

The polarity map: $V \times S^nV^* \xrightarrow{(v,f)\mapsto \mathrm{pl}_v f} S^{n-1}V^*$ is, clearly, bilinear in v, f and satisfies $\mathrm{pl}_v\mathrm{pl}_w f = \mathrm{pl}_w\mathrm{pl}_v f$ for any $v, w \in V$. So, *multiple polars*: $\mathrm{pl}_{v_1,v_2,\dots,v_m} f \stackrel{\mathrm{def}}{=} \mathrm{pl}_{v_1}\mathrm{pl}_{v_2} \cdots \mathrm{pl}_{v_1} f$ are symmetric and multilinear in v_i and linear in f. Hence, they come from the linear map $S^mV \otimes S^nV^* \longrightarrow S^{n-m}V^*$, which sends $v_1v_2 \cdots v_m \otimes f$ to the polynomial $\mathrm{pl}_{v_1,v_2,\dots,v_m} f(w) = \widetilde{f}(\, v_1, v_2, \dots, v_m\,, w, w, \dots, w)$. This map is called *a contraction of (symmetric) polynomials* (comp. with the last sections of 'Tensor Guide', especially with Exercice 4.29).

Partial derivatives. If some base $x_1, x_2, \dots, x_d$ in V^* is fixed and $\nu_i = x_i(v)$ are the coordinates of a vector $v \in V$, then we write $\dfrac{\partial f}{\partial v}$ for the derivative of f in the v-direction, i. e. for $\displaystyle\sum_{i=1}^{d} \nu_i \frac{\partial f}{\partial x_i}$.

Claim 6.1. $\dfrac{\partial f}{\partial v} = \deg(f) \cdot \mathrm{pl}_v f$

Proof. Let $\{e_i\}$ be base of V dual to $\{x_i\}$. By the linearity of $\mathrm{pl}_v f$ in v and f and indices symmetry reasons, it is enough to check that $\mathrm{pl}_{e_1} x_1^{m_1}x_2^{m_2} \cdots x_d^{m_d} = \dfrac{1}{n}\dfrac{\partial}{\partial x_1} x_1^{m_1}x_2^{m_2} \cdots x_d^{m_d}$ for all monomials with $\sum m_\nu = n = \deg(f)$. As we know (see eq. (24) in 'Tensor Guide'),

$$\mathrm{pl}\,(x_1^{m_1}x_2^{m_2} \cdots x_d^{m_d}) = \frac{m_1!\,m_2! \cdots m_d!}{n!} \cdot x_{[m_1,m_2,\dots,m_d]}\ ,$$

where $x_{[m_1,m_2,\dots,m_d]} \in \mathsf{ST}^nV^*$ is the sum of all different tensor products containing m_1 items x_1, m_2 items x_2, ..., m_d items x_d. The contraction with e_1, say in the first index, kills all terms which do not have e_1 as the first item and removes the first e_1 from all other terms. Hence, $\langle\, x_{[m_1,m_2,\dots,m_d]}\,,\, e_1 \,\rangle = x_{[(m_1-1),m_2,m_3,\dots,m_d]}$ and $\langle\, \mathrm{pl}\,(x_1^{m_1}x_2^{m_2} \cdots x_d^{m_d})\,,\, e_1 \,\rangle = \dfrac{m_1}{n}\,\mathrm{pl}\,(x_1^{m_1-1}x_2^{m_2}x_3^{m_3} \cdots x_d^{m_d}) = \dfrac{1}{n}\dfrac{\partial}{\partial x_1} x_1^{m_1}x_2^{m_2} \cdots x_d^{m_d}$. □

Exercice 6.2. Show that $\widetilde{f}(v_1, v_2, \dots, v_n) = \dfrac{1}{n!}\dfrac{\partial}{\partial v_1}\dfrac{\partial}{\partial v_2} \cdots \dfrac{\partial}{\partial v_n} f$ for any $f \in S^nV^*$ and any $v_1, v_2, \dots, v_n \in V$.

Taylor formula. The previous claim shows that $\dfrac{\partial f}{\partial v}$ does not depend on a base choice and satisfies the following symmetry properties:

$$m!\,\frac{\partial^m f}{\partial v^m}(w) = n!\,\widetilde{f}(\underbrace{v, v, \dots, v}_{m}, \underbrace{w, w, \dots, w}_{n}) = (n-m)!\,\frac{\partial^{n-m} f}{\partial w^{n-m}}(v) \tag{29}$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

where the both sides are bihomogeneous of degree $(m, n-m)$ in (v, w). This immediately implies *a Taylor formula*:

$$f(v+w) = \sum_{\nu=0}^{\deg f} \frac{1}{\nu!} \frac{\partial^\nu f}{\partial w^\nu}(v) \tag{30}$$

valid for any polynomial f over any field of zero characteristic. Indeed, we can assume f to be homogeneous, say $f \in S^n V^*$; then $f(v+w)$ is its full contraction with

$$(v+w)^m = \sum_m \binom{n}{m} v^m w^{n-m} \in S^n V \ .$$

So, the Taylor formula is nothing more than Newton's binomial expansion written in partial derivatives notations.

Linear span of a symmetric polynomial. A linear span $span(f)$ of a symmetric polynomial $f \in S^n V^*$ is a minimal subspace $W \subset V^*$ such that $f \in S^n W$. It coincides with a linear span of the full polarization $\mathrm{pl}(f) \in \mathsf{ST}^n V^*$ and, by Claim 4.11, with an image of the contraction map $S^{n-1}V \otimes S^n V^* \longrightarrow V^*$. In terms of partial derivatives this looks as

Corollary 6.3. *$span(f)$ is spanned by the linear forms*

$$\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{n-1}}} f(x) \tag{31}$$

obtained from f by all possible $(n-1)$-fold differenciations. □

Veronese variety in $\mathbb{P}_N = \mathbb{P}(S^n V^*)$, where $N = \binom{n}{d} - 1$ and $\dim V = d+1$, consists of pure n-th powers of linear forms, that is, of $f(x) = \xi^n(x)$, where $\xi \in \mathbb{P}(V^*)$. It has a rational parameterization given by the Veronese map

$$\mathbb{P}(V^*) \xhookrightarrow{\xi \longmapsto \xi^n} \mathbb{P}(S^n V^*) \ .$$

The previous corollary presents the Veronese variety by a system of quadratic equations. Namely, $f(x) = \xi^n(x)$ for some $\xi \in V^*$ iff $span(f)$ is 1-dimensional subspace generated by ξ. So, f is pure n-th power iff any two linear forms (31) are proportional to each other. If we arrange the coefficients of two linear forms as the rows of some $2 \times (d+1)$-matrix, then their proportionality means that all 2×2-minors of this matrix vanish.

Veronese's curve. Let $\dim U = 2$, $\mathbb{P}_1 = \mathbb{P}(U^*)$, $\mathbb{P}_n = \mathbb{P}(S^n U^*)$, $\{t_0, t_1\}$ be a base of U^*, and $\binom{n}{i} t_0^i t_1^{n-i}$ for $0 \leqslant i \leqslant n$ be the corresponding base of $S^n V^*$. Then the Veronese embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ sends a linear form $(\alpha_0 t_0 + \alpha_1 t_1)$ to

$$(\alpha_0 t_0 + \alpha_1 t_1)^n = \sum_i \alpha_0^i \alpha_1^{n-i} \cdot \binom{n}{i} t_0^i t_1^{n-i}$$

Its image is a rational curve $C_n \subset \mathbb{P}_n$ called *Veronese curve* or *rational normal curve* of degree n (comp. with Exercice 1.17) If we use the coefficients $(a_0 : a_1 : \ \ldots \ : a_n)$ and $(\alpha_0 : \alpha_1)$ as homogeneous coordinates for a polynomial $\sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i} \in S^n U^*$ and for a linear form $\alpha_0 t_0 + \alpha_1 t_1 \in \mathbb{P}(U^*)$, then C_n will be presented parametrically as $a_i = \alpha_0^i \alpha_1^{n-i}$.

On the other hand, for binary form $f(t) = \sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i}$ the linear forms (31) are exhausted by

$$\frac{\partial^i}{\partial t_0^i} \frac{\partial^{n-1-i}}{\partial t_1^{i-1-n}} f(t) = a_i t_0 + a_{i+1} t_1 \quad , \quad \text{where} \quad 0 \leqslant i \leqslant (n-1) \ .$$

So, $f \in C_n$ iff $\det \begin{pmatrix} a_i & a_{i+1} \\ a_{j-1} & a_j \end{pmatrix} = a_i a_j - a_{i+1} a_{j-1} = 0$ for all $0 \leqslant i < j \leqslant n$, i. e. iff

$$\mathrm{rk} \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} = 1 \ .$$

For $n = 2$ we get the Veronese quadric $a_0 a_2 = a_1^2$ in $\mathbb{P}_2$. For $n = 3$ we have a rational cubic curve, which is called *a twisted cubic*, given as an intersection of 3 quadrics $a_0 a_2 = a_1^2$, $a_1 a_3 = a_2^2$, $a_0 a_3 = a_1 a_2$.

Exercice 6.4. Draw the picture and discover that the first two quadrics are simple cones with vertices at $(0:0:0:1)$ and at $(1:0:0:0)$; the third quadric is Segre's one and is ruled by two line families; two cones have a common line element $a_1 = a_2 = 0$, which joins the vertices but does not lie on the Segre quadric; the line element $a_0 = a_1 = 0$ of the first cone and the one $a_2 = a_3 = 0$ of the second do lie on the Segre quadric in the same ruling family. So, any two of 3 quadrics are intersected along the twisted cubic and one more line, that is, the Veronese curve can not be given by use of 2 equations only!

LINEAR SPAN OF VERONESE CURVE. Given an arbitrary hyperplane $\pi = \{a \in \mathbb{P}_n \mid \sum A_i a_i = 0\}$, the intersection $C_n \cap \pi$ consists of all $a = a(\alpha)$ that satisfy the equation $\sum_i A_i \alpha_0^i \alpha_1^{n-i} = 0$ whose left side is non zero polynomial of degree n. So, a hyperplane section of Veronese's curve always consists[2] of n points counted with appropriate multiplicities (typically, of n *distinct* points). In particular, the linear span of any $(n+1)$ Veronese's curve points gives the whole of $\mathbb{P}_n$.

Corollary 6.5. [3] *To prove that some linear in f assertion holds for all polynomials f, it is enough to verify it only for all powers of all linear forms.* □

Exercice 6.6. Use Arongold principle to give another proof for Taylor formula (30).

Exercice 6.7. Let $p_0, p_1, \dots, p_n \in S^n U^*$ be any linearly independent polynomials. Show that the parameterized curve $\alpha \longmapsto (p_0(\alpha) : p_1(\alpha) : \dots\ p_n(\alpha))$ turns into Veronese's curve under an appropriate linear automorphism of $\mathbb{P}_n$.
HINT. a matrix of the linear transformation in question is inverse to the matrix whose rows are the coefficients of p_i.

Example 6.8: Projecting twisted cubic. Let us describe all plane projections of the twisted cubic $C_3 \subset \mathbb{P}_3 = \mathbb{P}(S^3 U^*)$. Up to a projective isomorphism, the projection does not depend on the choice of a target plane as soon as the center is fixed, because the projection of one target plane onto another gives an linear isomorphism between the projection images. Let $p = p(t) = \ell_1(t)\ell_2(t)\ell_3(t) \in \mathbb{P}_3 = \mathbb{P}(S^3 V^*)$ be a projection center. After some parameter change we can suppose that either $\ell_1 = \ell_2 = \ell_3 = t_0^2$, (this means that $p \in C$), or $\ell_1 = \ell_2 = t_0$, $\ell_3 = t_1$ ([4]), or $\ell_1 = (t_0 + t_1)$, $\ell_2 = (t_0 + \omega\, t_1)$, $\ell_3 = (t_0 + \omega^2 t_1)$, where $\omega = \sqrt[3]{1} \neq 1$ (that is, $p(t) = t_0^3 + t_1^3$ has 3 distinct roots).

In the first case $p = (1:0:0:0)$; take a target plane to be $a_0 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_1 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^2 : \alpha_0\alpha_1 : \alpha_1^2)$ and coincides with the plane Veronese conic $x_0 x_2 = x_1^2$.

In the second case $p = (0:1:0:0)$; take a target plane to be $a_1 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_0 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^3 : \alpha_0\alpha_1^2 : \alpha_1^3)$ and in the affine chart $\{x_0 = 1\}$ it turns into $x = \alpha^2$, $y = \alpha^3$, where $x = x_1/x_0$, $y = x_2/x_0$, and $\alpha = \alpha_1/\alpha_0$. So, we get a curve $y^2 = x^3$ or, in the homogeneous coordinates, $x_2^3 = x_3^2 x_0$. This curve is called *a cuspidal cubic*, because of the singularity form at the origin.

In the third case $p = (1:0:0:1)$; take a target plane $\pi = \{a_2 = 0\}$ with the coordinates $(x_0 : x_1 : x_2) = ((a_0 - a_1) : a_1 : a_2)$ (the first three coordinates w.r.t. the base $\{\, t_0^3, 3\,t_0^2 t_1\,,\ 3\,t_0 t_1^2\,,\ t_0^3 + t_1^3 \,\}$). So, the projection from $p = t_0^3 + t_1^3$ gives the parameterized curve $(x_0 : x_1 : x_2) = \big((\alpha_0^3 - \alpha_1^3) : \alpha_0^2\alpha_1 : \alpha_0\alpha_1^2\big)$. In affine chart $x_0 = 1$ we get, like above, $x = \alpha/(1-\alpha^3)$, $y = \alpha^2/(1-\alpha^3)$ or $xy = x^3 - y^3$. This curve has a self intersection point at the origin and is called *a nodal* cubic.

GEOMETRIC DESCRIPTION OF RATIONAL CURVES. A plane curve $C \subset \mathbb{P}_2$ is called *rational* if there are 3 coprime homogeneous polynomials $p_0(t)$, $p_1(t)$, $p_2(t)$ of the same degree in $t = (t_0 : t_1)$ such that a map $\mathbb{P}_1 \xrightarrow{\alpha \mapsto (p_0(\alpha) : p_1(\alpha) : p_2(\alpha))} \mathbb{P}_2$ gives (maybe, after removing some finite sets of points from $\mathbb{P}_1$ and C) a bijection between $\mathbb{P}_1$ and C .

Exercice 6.9. Intersecting C with lines, show that $\deg C = \deg p_i$.
When $\deg p_i = d$, a map $(t_0^d : t_0^{d-1}t_1 : \dots : t_0 t_1^{d-1} : t_1^d) \longmapsto (p_0(t) : p_1(t) : p_2(t))$ defines a projection of the Veronese curve $C_d \subset \mathbb{P}_d$ into some plane $\mathbb{P}_2 \subset \mathbb{P}_d$. So, we have

Claim 6.10. *Each rational plane curve of degree d is an appropriate projection of the Veronese curve $C_d \subset \mathbb{P}_d$.* □

Corollary 6.11. *A smooth plane cubic curve is not rational.*

Proof. Rational cubic curve is a plane projection of the twisted cubic $C_3 \subset \mathbb{P}_3$. But such a projection is either a conic or a singular cubic. □

SIMPLICES INSCRIBED INTO VERONESE CURVE. For $1 \leqslant i \leqslant n$ let $p_i = \xi_i^n$ be any n distinct points on the Veronese curve $C_n \subset \mathbb{P}_n = \mathbb{P}(S^n U^*)$. For each i consider the pencil of hyperplanes passing through $(n-2)$-dimensional face $(p_1, \dots\ p_{i-1}, p_{i+1}, \dots, p_n)$ opposite to p_i in the $(n-1)$ dimensional simplex $(p_1, p_2, \dots, p_n)$. These n pencils are parameterized uniformly by the points of $\mathbb{P}_1^\times = \mathbb{P}(U)$ as follows. For any $\xi \in \mathbb{P}(U^*)$ denote by $\widehat{\xi} \in \mathbb{P}(U)$ the annihilator[5] $\mathrm{Ann}\,(\xi)$ and for each i take the product $\tau_i = \widehat{\xi}_1, \cdots, \widehat{\xi}_{i-1}\widehat{\xi}_{i+1}, \cdots, \widehat{\xi}_1 \in S^{n-1}U$. Define a plane

[2] recall that we suppose the ground field to be algebraically closed
[3] it is known as *Arongold principle*
[4] geometrically, this means that p lies on a tangent line to C_3
[5] if $\xi = \alpha_0 t_0 + \alpha_1 t_1$, then $\widehat{\xi} = \alpha_1 e_0 - \alpha_0 e_1$, where $\{e_o, e_1\} \subset U$ is the base of U dual to $\{t_0, t_1\} \subset U^*$

$\pi_i(u) \subset \mathbb{P}(S^nU^*)$, which corresponds to $u \in \mathbb{P}(U)$ in i-th pencil, as the annihilator of $u\tau_i \in S^nU$. This means that

$$\pi_i(u) = \left\{ f(t) \in S^nU^* \mid \widetilde{f}(\widehat{\xi}_1, \ldots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \ldots, \widehat{\xi}_n) = 0 \right\}$$

where $\widetilde{f}$ is the full polarization of f considered as a symmetric multilinear form on U. In particular, for $f(t) = \zeta^n(t) \in C_n$ we have

$$(\zeta^n)^{\sim}(\widehat{\xi}_1, \ldots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \ldots, \widehat{\xi}_n) = \zeta(u) \prod_{\nu \neq i} \zeta\left(\widehat{\xi}_\nu\right)$$

So, for any $u \in \mathbb{P}(U)$ the plane $\pi_i(u)$ pass through all $p_\nu = \xi_\nu^n$ with $\nu \neq i$ and through the point $p = \xi^n \in C_n$ whose ξ annihilate u (i. e. such that $u = \widehat{\xi}$). In other words,

$$C_n = \bigcup_u \pi_1(u) \cap \pi_2(u) \cap \cdots \cap \pi_n(u) \tag{32}$$

Since a linear isomorphism between projective lines is uniquely defined by the images of any 3 distinct points, we get the following corollary.

Claim 6.12. *The Veronese curve is uniquely recovered from any its $(n+3)$ distinct points a, b, c, $p_1, p_2, \ldots, p_n$ as follows. Consider n hyperplane pencils through the $(n-2)$-dimensional faces of the inscribed simplex $(p_1, p_2, \ldots, p_n)$ and parameterize them uniformly by $u \in \mathbb{P}_1$ in such a way that the hyperplanes passing through a, b, c appear in each pencil when $u = 0, 1, \infty$. Then C_n coincides with the incidence graph (32), where u runs through the parameter line $\mathbb{P}_1$.* □

Projective linear group action. The group $PGL_2(k)$ acts on $\mathbb{P}(S^nU^*)$ via linear variable changes $(t_0, t_1) \longmapsto (at_0 + bt_1, ct_0 + dt_1)$. Projective automorphisms $\mathbb{P}(S^nU^*) \xrightarrow{\sim} \mathbb{P}(S^nU^*)$ which are induced by these substitutions send the Veronese curve to itself. We call them *reparameterizations* of the Veronese curve.

Claim 6.13. *Let $p_1, p_2, \ldots, p_n, a, b, c \in \mathbb{P}_n = \mathbb{P}(S^nU^*)$ be any $n+3$ points with no $(n+1)$ on the same hyperplane. Then there exists a projective linear isomorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ that sends these points onto Veronese curve C_n; this isomorphism is unique up to a reparameterization of the Veronese curve.*

Proof. For each $i = 1, 2, \ldots, n$ identify $\mathbb{P}_1 = \mathbb{P}(U)$ with a pencil of hyperplanes through $p_1, \ldots p_{i-1}, p_{i+1}, \ldots, p_n$ by sending $u = e_0$, e_1, $(e_1 - e_0)$ to the hyperplanes that contain a, b, c and denote by $\pi_i(u)$ the u-th hyperplane in the i-pencil. Let the hyperplane $(p_1, p_2, \ldots, p_n)$ appear in i-th pencil when $u = u_i$. Then $u_1, u_2, \ldots, u_n \in \mathbb{P}(U)$ are mutually distinct by the following reason. Consider 2-dimensional plane $\Pi = (a, b, c)$ and denote by q_1 and q_2 its intersection with i-th and j-th $(n-2)$-dimensional faces of $(p_1, p_2, \ldots, p_n)$. Sitting in this plane, we will see the picture shown on 6.1. Our i-th and j-th pencils of hyperplanes are represented inside Π by the pencils of lines passing through q_1, q_2 and the hyperplane $(p_1, p_2, \ldots, p_n)$ is represented by the line $(q_1 q_2)$ (compare this construction with drawings from the **lecture 2**).

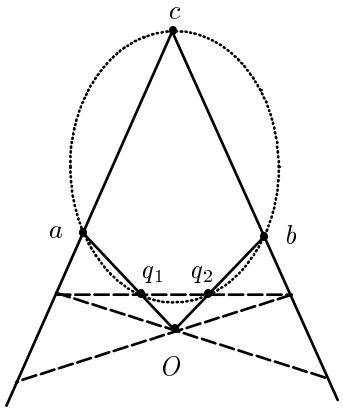

Fig. 6.1. Comparing u_i and u_j.

Exercice 6.14. Show that no 3 of q_1, q_2, a, b, c are collinear (in particular, q_1, q_2 are distinct).

Hint. In the contrary case some $(n+1)$ of $p_1, p_2, \ldots, p_n, a, b, c$ would lie on a hyperplane

There are two ways to identify the parameter line $\mathbb{P}(U)$ with the pencil of lines passing through $O = (aq_1) \cap (bp_2)$: we can send u either to the line through $\pi_i(u) \cap (bc)$ or to the line through $\pi_j(u) \cap (ac)$. These parameterizations should coincide, because they attach the same u to the lines through a, b, c. The lines which correspond to $u = u_i$ and $u = u_j$ join O with $\pi_i(u_i) \cap (bc) = (q_1 q_2) \cap (bc)$ and with $\pi_j(u_j) \cap (ac) = (q_1 q_2) \cap (ac)$. Since O, q_1 and q_2 are not collinear, these 2 lines are distinct, i. e. $u_i \neq u_j$.

Now, denote by Γ the incidence graph (32) build from our current pencils of hyperplanes. Let $u_i = \mathrm{Ann}\,(\xi_i)$ for $\xi_1, \xi_2, \ldots, \xi_n \in U^*$. By the Lemma 1.11, there exists a unique projective linear automorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ which sends $p_i \longmapsto \xi_i^n$ for $1 \leqslant i \leqslant n$, $a \longmapsto t_1^n$, and $b \longmapsto t_0^n$. It identifies Γ with the Veronese curve, because it sends the hyperplane pencil through $p_1, \ldots p_{i-1}, p_{i+1}, \ldots, p_n$ to the one through $\xi_1^n, \ldots \xi_{i-1}^n, \xi_{i+1}^n, \xi_n^n$ in such a way that $\pi_i(u)$ goes to the hyperplane through ξ^n as soon as $\xi = \mathrm{Ann}\, u$. Indeed, this takes place for $u = e_1$, $u = e_0$ and $u = u_i$ when a, b and p_i go to t_0^n, t_1^n and ξ_i^n. Hence, this holds for each u and for $u = e_1 - e_0$ we have $c \longmapsto (t_0 + t_1)^n$. This proves the existence.

Uniqueness follows from the above construction as well. Namely, after appropriate parameter change we can suppose that the isomorphism in question sends a, b, c to t_0^n, t_1^n and $(t_0 + t_1)^n$. So, it induces the uniform parameterization of hyperplane pencils trough p_ν and this parameterization coincides with the above one. So, the images of $p_1, p_2, \ldots, p_n$ are uniquely recovered as $\xi_i = \mathrm{Ann}\,(u_i)$. □