Algebraic – geometric dictionary.

AFFINE ALGEBRAIC SETS AND THEIR IDEALS. Let $X \subset \mathbb{A}^n$ be an affine algebraic set given by a system of polynomial equations $f_\nu(x_1, x_2, \dots, x_n) = 0$. Then $X = V(J)$, where $J = \subset k[x_1, x_2, \dots, x_n]$ is an ideal generated by f_ν (algebraically, J is obtained by adding to the initial equation system all its consequences $\sum g_\nu f_\nu = 0$ with $g_\nu \in k[x_1, x_2, \dots, x_n]$). We write $I(X)$ for the ideal of all polynomials vanishing along X. If k is algebraically closed and $X = V(J)$, then $I(X) = \sqrt{J} \stackrel{\text{def}}{=} \{ f \in k[x_1, x_2, \dots, x_n] \mid f^n \in J \text{ for some } n \in \mathbb{N} \}$ is *a radical* of J by Hilbert's Nullstellensatz. Clearly, $V(J_1) \subset V(J_2) \iff \sqrt{J_1} \supset \sqrt{J_2}$.

Example 11.1: one point sets. An affine algebraic set $V(I) \subset \mathbb{A}(V)$ coincides with the origin $\{O\} \in \mathbb{A}(V)$ iff $I \supset S^d V^*$ for all $d \gg 0$ and $1 \notin I$. Indeed, for each i some power $x_i^{m_i} \in I$, because each x_i vanishes at O. So, $S^d V^* \subset I$ for $d \geqslant 1 + \sum (m_\nu - 1)$. Conversely, $V(I) = \{O\}$, if I contains some powers $x_i^{m_i}$ of all coordinate functions.

Exercice 11.2. Prove that $I(\{O\})$ is a maximal ideal and is generated by x_i.

Example 11.3: resultant systems. We fix a collection of $(n+1)$ degrees $d_0, d_1, \dots, d_n$ and write $\mathcal{S}_d = \mathbb{P}(S^d V^*)$ for the space of hypersurfaces of degree d in $\mathbb{P}_n = \mathbb{P}(V)$. Let $\mathcal{R} \subset \mathcal{S}_{d_0} \times \mathcal{S}_{d_1} \times \dots \times \mathcal{S}_{d_n}$ be a set of all hypersurface collections $S_0, S_1, \dots, S_n \subset \mathbb{P}_n$ such that $\bigcap S_\nu \neq \varnothing$. Then $\mathcal{R}$ is a projective algebraic set, i. e. can be described by a finite system of multi-homogeneous polynomial equations on the coefficients of forms $(f_0, f_1, \dots, f_n) \in S^{d_0} V^* \times S^{d_1} V^* \times \dots \times S^{d_n} V^*$ presenting the hypersurfaces $S_0, S_1, \dots, S_n$ (this equation system depends on n, $d_0, d_1, \dots, d_n$ and is called *a resultant system*). Indeed, consider an ideal $I \subset k[x_0, x_1, \dots, x_n]$ generated by f_ν; then $\bigcap S_\nu \subset \mathbb{P}(V)$ is empty $\iff V(I) \subset \mathbb{A}(V)$ either is empty or coincides with the origin $\iff S^d V^* \subset I \ \forall\, d \gg 0 \iff k$-linear map:

$$\mu_d \ : \ S^{d-d_0} V^* \oplus S^{d-d_1} V^* \oplus \dots \oplus S^{d-d_n} V^* \xrightarrow{(g_0, g_1, \dots, g_n) \mapsto \sum g_\nu f_\nu} S^d \tag{38}$$

is not surjective $\forall\, d \gg 0$. In terms of the standard monomial bases, μ_d is presented by a matrix whose entries are linear forms in the coefficients of f_ν. Since for $d \gg 0$ the dimensions of vector spaces in the left and right sides of (38) satisfy inequality $\sum\limits_{\nu=0}^{n} \binom{n+d-d_\nu}{n} \geqslant \binom{n+d}{n}$, we can describe $\mathcal{R}$ as the zero set of all maximal minors of all matrices μ_d with d large enough. By Hilbert's base theorem, this infinite equation system is equivalent to some finite subsystem. For example, for $n = 1$ the smallest d producing a non trivial restriction is $d = d_0 + d_1$, where μ_d becomes a square matrix. As we know from Claim 7.29 the equation $\det \mu_d = 0$ for this d implies all the further: two binary forms of degrees d_0, d_1 have a common zero on $\mathbb{P}_1$ iff their resultant vanishes. So, $\mathcal{R}$ is a hypersurface for $n = 1$.

Example 11.4: projection $\mathbb{P}_m \times \mathbb{A}^n \xrightarrow{\pi} \mathbb{A}^n$ sends algebraic sets to algebraic sets, i. e. $X = \{ (q, p) \in \mathbb{P}_m \times \mathbb{A}^n \mid f_\nu(q, p) = 0 \}$ given by polynomial equations $f_\nu(t, x) = 0$ (homogeneous in $t = (t_1, t_2, \dots, t_m) \in \mathbb{P}_m$) is projected onto some affine algebraic subset in $\mathbb{A}^n$. Indeed, consider f_ν as homogeneous polynomials in t with the coefficients in $k[x_1, x_2, \dots, x_n]$. Then the image $\pi(X) \subset \mathbb{A}^n$ consists of all p such that $f_\nu(t, p)$ have a common zero on $\mathbb{P}_m$. By the Example 11.3, this means that their coefficients satisfy the resultant equation system. So, such p form an algebraic set.

COORDINATE ALGEBRA. Finitely generated commutative k-algebra $k[X] \stackrel{\text{def}}{=} k[x_1, x_2, \dots, x_n]/I(X)$ is called *a coordinate algebra* of affine algebraic set $X \subset \mathbb{A}^n$. By Hilbert's Nullstellenzatz it is reduced[1] as soon k is algebraically closed. Geometrically, $k[X]$ consists of functions $X \xrightarrow{v \mapsto f(v)} k$ obtained by restricting the polynomials $f \in k[x_1, x_2, \dots, x_n]$ onto $X \subset \mathbb{A}^n$; these functions are called *regular algebraic functions* on X.

Claim 11.5. *Let A be a finitely generated algebra over an algebraically closed field. Then $A = k[X]$ for some affine algebraic set X iff A is reduced.*

Proof. Any algebra of field-valued functions is obviously reduced. Conversely, if a finitely generated k-algebra $A = k[x_1, x_2, \dots, x_n]/I$ is reduced, then $f^n \in I \Rightarrow f \in I$ for any $f \in k[x_1, x_2, \dots, x_n]$. By Hilbert, this means that $X = V(I) \subset \mathbb{A}^n$ has $I(X) = I$. So, $A = k[x_1, x_2, \dots, x_n]/I(X) = k[X]$. □

POINTS. Given a point $p \in X$, we call $\mathfrak{m}_p \stackrel{\text{def}}{=} \{ f \in k[X] \mid f(p) = 0 \}$ *a maximal ideal* of p. Since the evaluation map $\mathrm{ev}_p : k[X] \xrightarrow{f \mapsto f(p)} k$ coincides with the factorization $k[X] \xrightarrow{f \mapsto f \,(\mathrm{mod}\, \mathfrak{m}_p)} k[X]/\mathfrak{m}_p$, we have $k[X]/\mathfrak{m}_p = k$, i. e. $\mathfrak{m}_p$ really is a maximal proper ideal of $k[X]$.

Claim 11.6. *If k is algebraically closed, then the correspondences $p \longleftrightarrow \mathrm{ev}_p \longleftrightarrow \mathfrak{m}_p \stackrel{\text{def}}{=} \ker(\mathrm{ev}_p)$ establish bijections between the points of X, the homomorphisms $k[X] \longrightarrow k$ identical on k, and the proper maximal ideals in $k[X]$.*

Proof. The second correspondence is obviously bijective. Clearly, $p \neq q \Rightarrow \mathfrak{m}_p \neq \mathfrak{m}_q$, because there is, say, a linear form $\mathbb{A}^n \xrightarrow{\ell} k$ such that $\ell \in \mathfrak{m}_p$ but $\ell \notin \mathfrak{m}_q$. Finally, consider a proper maximal ideal $\mathfrak{m} \subset k[X]$ and write $k[X]$ as

[1] i. e. does not contain nilpotents: $a^n = 0 \Rightarrow a = 0$

$k[x_1,x_2,\dots,x_n]/I(X)$. A full preimage $\widetilde{\mathfrak{m}}\subset k[x_1,x_2,\dots,x_n]$ of $\mathfrak{m}$ is a proper maximal ideal, because of $\widetilde{\mathfrak{m}}\supset I(X)$ and $k[x_1,x_2,\dots,x_n]/\widetilde{\mathfrak{m}}=k[X]/\mathfrak{m}=k$. Hence, $V(\widetilde{\mathfrak{m}})\subset\mathbb{A}^n$ is nonempty and is contained in X, i. e. there is a point $p\in X$ such that $f(p)=0$ for any $f\in\mathfrak{m}$. So, $\mathfrak{m}\subset\mathfrak{m}_p$ and $\mathfrak{m}=\mathfrak{m}_p$, because $\mathfrak{m}$ is maximal. □

Algebraic sets via spectra. A set of all proper maximal ideals in a given k-algebra A is called *a maximal spectrum* of A and is denoted by $Spec_m(A)$. In pure algebraic (and coordinateless) terms, an affine algebraic set over an algebraically closed field k is a maximal spectrum $Spec_m(A)$ of a finitely generated reduced k-algebra A whose elements $f\in A$ are considered as the functions $Spec_m(A)\xrightarrow{\mathfrak{m}\mapsto f\,(\mathrm{mod}\,\mathfrak{m})}k$.

Regular morphisms of algebraic sets. Any map of sets $X\xrightarrow{\varphi}Y$ produces *a pull back homomorphism* φ^* from the algebra off all k-valued functions on Y to the one on X; it sends $Y\xrightarrow{f}k$ to the composition $\varphi^*f: X\xrightarrow{\varphi}Y\xrightarrow{f}k$. A map $X\xrightarrow{\varphi}Y$ between affine algebraic sets is called *a regular morphism* of algebraic sets, if φ^* sends the regular algebraic functions on Y to the regular algebraic functions on X, i. e. induces a well defined homomorphism of coordinate algebras $k[Y]\xrightarrow{\varphi^*}k[X]$.

Claim 11.7. *Let $B\xrightarrow{\varphi^*}A$ be a homomorphism of finitely generated reduced algebras over algebraically closed field k such that $\varphi^*(1)=1$. Then φ^* is a pull back homomorphism for a unique regular map $Spec_m(A)\xrightarrow{\varphi}Spec_m(B)$; this map sends a maximal ideal $\mathfrak{m}\subset A$ to its full preimage $\varphi^{*-1}(\mathfrak{m})\subset B$.*

Proof. A pull back homomorphism of a regular morphism $Spec_m(A)\xrightarrow{\varphi}Spec_m(B)$ takes $\varphi^*(\mathfrak{m}_{\varphi(p)})=\mathfrak{m}_p$ for any $p\in Spec_m(A)$, because of $f(\varphi(p))=0\iff\varphi^*f(p)=0$. So, if φ exists, then it has to send $\mathfrak{m}\mapsto\varphi^{*-1}(\mathfrak{m})$. On the other hand, this rule really presents a regular morphism. Indeed, $\varphi^*(1)=1$ implies that $\varphi^{*-1}(\mathfrak{m})\subset B$ is a proper maximal ideal, because of $B/\varphi^{*-1}(\mathfrak{m})=\mathrm{im}\,(\varphi^*)/(\mathfrak{m}\cap\mathrm{im}\,(\varphi^*))\simeq k$. So, φ is well defined map of sets. Its pull back homomorphism sends a function $b\in B$, which takes a maximal ideal $\mathfrak{q}\in B$ to $b\,(\mathrm{mod}\,\mathfrak{q})\in k$, to the function $\mathfrak{m}\longmapsto b\,(\mathrm{mod}\,\varphi^{*-1}(\mathfrak{m}))=\varphi^*(b)\,(\mathrm{mod}\,\mathfrak{m})\in k$, which coincides with the function $\varphi^*(b)$ on $Spec_m(A)$. □

Geometric schemes. Let A be an arbitrary algebra over a field k. The set of all nilpotent elements $\mathfrak{n}(A)\subset A$ is an ideal contained in any proper maximal ideal $\mathfrak{m}\subset A$, because $A/\mathfrak{m}=k$ has no nilpotent elements. So, a factor algebra $A_{\mathrm{red}}\stackrel{\mathrm{def}}{=}A/\mathfrak{n}$ is reduced and has the same maximal spectrum $X=Spec_m A=Spec_m A_{\mathrm{red}}$. If k is algebraically closed, then the intersection of all maximal ideals in A_{red} is zero, because it consists of all functions vanishing everywhere on the affine algebraic set X. Hence, the intersection of all maximal ideals in A coincides with $\mathfrak{n}(A)$. A maximal spectrum $Spec_m A$ of an arbitrary finitely generated algebra A over an algebraically closed field is called *an affine geometrical scheme*. Intuitively, this is an affine algebraic set $X=Spec_m A_{\mathrm{red}}$ equipped with some extra «nilpotent functions» vanishing everywhere on X. Usually, these nilpotent elements encode some infinitesimal data on X.

Example 11.8: an intersection of affine algebraic sets $X,Y\subset\mathbb{A}^n$ is defined as $V(I(X)+J(X))$, i. e. by the union of all equations for X and Y. If the intersection is non-transversal, a factor algebra $A=k[x_1,x_2,\dots,x_n]/(I(X)+J(X))$ is not reduced. Of course, the intersection points are exhausted by an algebraic set $Spec_m A_{\mathrm{red}}$, but a significant information will be lost under the replacement of A by A_{red}. If we want, say, to develop an intersection multiplicities technique, then we have to treat intersections as *schemes* rather than as algebraic sets and investigate the difference between A and A_{red}.

A direct product of affine algebraic sets. Let A, B be finitely generated k-algebras. We write $A\otimes B$ for its tensor product as vector spaces[2] over k. In fact, $A\otimes B$ is an finitely generated k-algebra with the multiplication map $(A\otimes B)\times(A\otimes B)\xrightarrow{(a\otimes b,\alpha\otimes\beta)\mapsto a\alpha\otimes b\beta}A\otimes B$.

Exercice 11.9. Using the universality of the tensor product, verify the correctness of an algebra structure definition and prove that k-algebra homomorphisms $A\xrightarrow{\alpha:a\mapsto a\otimes 1}A\otimes B\xleftarrow{\beta:b\mapsto 1\otimes b}B$ have the following universal property: for any two algebra homomorphisms $A\xleftarrow{\varphi}C\xrightarrow{\psi}B$ there is a unique homomorphism $C\xrightarrow{\varphi\otimes\psi}A\otimes B$ such that $\varphi\otimes\psi=\alpha\circ\varphi=\beta\circ\psi$.

Claim 11.10. *A product of affine algebraic sets $X=Spec_m A$, $Y=Spec_m B$ has a natural structure of an affine algebraic set given as $X\times Y=Spec_m(A\otimes B)$.*

Proof. On the level of sets, a bijection $Spec_m(A)\times Spec_m(B)\simeq Spec_m(A\otimes B)$ takes any point $(p,q)\in X\times Y$, that is a pair of epimorphisms $A\xrightarrow{\mathrm{ev}_p}k$, $B\xrightarrow{\mathrm{ev}_q}k$, or a bilinear map $A\times B\xrightarrow{(f,g)\mapsto f(p)g(q)}k$, to k-linear epimorphism $A\otimes B\longrightarrow k$, which is clearly multiplicative w. r. t. the algebra structure on $A\otimes B$. Wise versa, an algebra homomorphism $A\otimes B\xrightarrow{\psi}k$, which takes $1\otimes 1$ to 1, gives a pair of epimorphisms $A\xrightarrow{f\mapsto\psi(f\otimes 1)}k$, $B\xrightarrow{g\mapsto\psi(1\otimes g)}k$. To show that

[2] In general, a finitely generated k-algebra is an infinite dimensional vector space with countable base, that is, consists of all *finite* formal linear combinations $\sum\lambda_\nu e_\nu$ with e_ν belonging to some countable family of linearly independent vectors and $\lambda_\nu\in k$. It is easy to see that the finite dimensional constructions from **lecture 4** do work for such spaces as well. In particular, $A\otimes B$ consists of all *finite* sums $\sum a_\nu\otimes b_\nu$ with $a_\nu\in A$, $b_\nu\in B$ and is uniquely defined by the universal property discussed in **lecture 4**; for example, $k[x]\otimes k[y]$ is naturally isomorphic to $k[x,y]$.

$A \otimes B$ is a reduced, we write $\varphi \in A \otimes B$ as $\sum f_\nu \otimes g_\nu$ with linearly independent $g_\nu \in B$;, if it produces a zero function on $X \times Y$, then $\varphi(p,q) = 0\ \forall (p,q) \in X \times Y \Rightarrow \sum f_\nu(p)\, g_\nu = 0$ in $k[Y] \Rightarrow$ all $f_\nu = 0$ in $A \Rightarrow \varphi = 0$ in $A \otimes B$. □

Exercice 11.11. Check that the pull back homomorphisms for the projections $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ coincide with the maps α, β from Exercice 11.9 and prove that for any regular morphisms $X \xrightarrow{\alpha} Z$, $Y \xrightarrow{\beta} Z$ there exists a unique regular morphism $X \times Y \xrightarrow{\alpha \times \beta} Z$ such that $\alpha \times \beta = \alpha \circ \pi_1 = \beta \circ \pi_2$.

ZARISKI TOPOLOGY. Any affine algebraic set $X = Spec_m A$ admits a canonical topology whose closed sets are zeros of ideals $I \subset A$, i. e. the sets $V(I) = \{x \in X \mid f(x) = 0 \quad \forall f \in I\ \}$.

Exercice 11.12. Check that $V(I)$ satisfy the closed set properties, namely, $\varnothing = V(1)$; $X = V(0)$; $\bigcap V(I_\nu) = V\big(\sum I_\nu\big)$, where $\sum I_\nu$ consists of all finite sums $\sum f_\nu$ with $f_\nu \in I_\nu$; $V(I) \cup V(J) = V(IJ)$, where IJ is an ideal spanned by all products ab with $a \in I$, $b \in J$.

Since any ideal $I \subset k[X]$ is finitely generated, each closed set may be presented as a finite intersection of hypersurfaces $V(I) = V(f_1, f_2, \dots, f_m) = \bigcap V(f_\nu)$. So, any Zariski open set is a finite union of *principal* open sets $\mathcal{D}(f) \stackrel{\text{def}}{=} X \setminus V(f) = \{x \in X \mid f(x) \neq 0\}$. Zariski topology has a pure algebraic nature: its properties are far enough from the metric topology standards, because Zariski neighborhoods express rather some divisibility conditions than any distance relations.

Example 11.13: irreducible closed sets. A topological space X is called *reducible*, if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subset X$. It's a vapid notion for the usual metric topology, where everything is reducible. In Zariski topology, the reducibility means an existence of non zero $f_1, f_2 \in k[X]$ such that f_1 vanishes along X_1 and f_2 vanishes along X_2. Since $f_1 f_2$ vanishes everywhere, $f_1 f_2 = 0$ in $k[X]$. So, X is reducible iff $k[X]$ has zero divisors. For example, a hypersurface $\{g(x) = 0\} \subset \mathbb{A}^n$ is irreducible iff g is a power of an irreducible polynomial. Irreducible algebraic sets generalize the prime numbers in arithmetics.

Claim 11.14. *Any affine algebraic set admits a unique finite decomposition $X = \bigcup X_i$, where $X_i \subset X$ are irreducible proper closed subsets such that $X_i \not\subset X_j\ \forall\ i \neq j$ (they are called irreducible components of X).*

Proof. A decomposition is constructed step by step. If X is reducible, the first step takes $X = Z_1 \cup Z_2$, where $Z_{1,2}$ are proper closed subsets. Let $X = \bigcup Z_\nu$ after n steps. If each Z_ν is irreducible, then for each ν and any irreducible closed subset $Y \subset X$ either $Y \cap Z_\nu = \varnothing$ or $Y \subset Z_\nu$, because of $Y = \bigcup (Z_\nu \cap Y)$. So, if we take away all Z_ν contained in some other Z_μ, then we get the required decomposition and it is unique. If there are some reducible Z_ν after n steps, $(n+1)$-th step replaces each of them by an union of two proper closed subsets. If this procedure would never stop, then it produces an infinite chain of strictly embedded subsets $X \supset Y_1 \supset Y_2 \supset \dots$, i. e. an infinite chain of strictly increasing ideals $(0) \subset I_1 \subset I_2 \subset \dots$, which does not exist in the Noetherian algebra $k[X]$. □

Example 11.15: «big» open sets. Zariski topology is week and non Hausdorf. For example, Zariski topology on $\mathbb{A}^1$ is discrete: $Z \subset \mathbb{A}^1$ is closed iff it is finite. If X is irreducible, then any open $U_1, U_2 \subset X$ have nonempty intersection, because of $X = (X \setminus U_1) \cup (X \setminus U_2)$ in the contrary case.

Exercice 11.16. Prove that $f = g$ in $k[X]$, if X is irreducible and $f|_U = g|_U$ over some open non-empty $U \subset X$.
HINT. If $U = \mathcal{D}(h)$, then $X = V(h) \cup V(f - g)$.

Example 11.17: Zariski topology on $X \times Y$ is finer than the product of Zariski topologies on X and Y, because the closed $Z \subset X \times Y$ are not exhausted by the products of closed subsets on X, Y. For example, if $X = Y = \mathbb{A}^1$, then any curve, say a hyperbola $V(xy - 1)$, is closed in Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, whereas the products of closed sets on $\mathbb{A}^1$ are exhausted by finite unions of isolated points and coordinate lines.

Claim 11.18. *A regular morphism $X \xrightarrow{\varphi} Y$ of algebraic sets is continuous in Zariski topology.*

Proof. A preimage $\varphi^{-1}(Z)$ of a closed $Z = V(I) \subset Y$ consists of all $x \in X$ such that $0 = f(\varphi(x)) = \varphi^{*-1} f(x)$ for all $f \in I$. So, it coincides with the zero set of an ideal $\varphi^{*-1}(I) \subset k[X]$. □

MORPHISM DECOMPOSITION. A regular morphism of affine algebraic sets $X \xrightarrow{\varphi} Y$ defines and is uniquely defined by the k-algebra homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$, which is decomposed as $k[Y] \twoheadrightarrow \operatorname{im}(\varphi^*) \hookrightarrow k[X]$. Since $k[X]$ is reduced, $\operatorname{im}(\varphi^*) \subset k[X]$ is too. So, there is an affine algebraic set $Z = Spec_m \operatorname{im}(\varphi^*)$ such that $X \xrightarrow{\varphi} Y$ is decomposed as $X \xrightarrow{\varphi_1} Z \xrightarrow{\varphi_2} Y$, where $k[Y] \xrightarrow{\varphi_2^*} k[Z]$ is surjective, and $k[Z] \xrightarrow{\varphi_1^*} k[X]$ is injective. The injectivity of φ_1 means that there are no non zero functions $f \in k[Z]$ vanishing along $\varphi_1(X)$, i. e. $\varphi_1(X) \subset Z$ is dense. The surjectivity of φ_2^* means that φ_2 induces an isomorphism between Z and a closed subset $V(\ker \varphi_2^*) \subset Y$ given by the ideal $\ker(\varphi_2^*) = \ker(\varphi) \subset k[Y]$ (which is automatically radical). In other words, the Zariski closure $Z = \overline{\varphi(X)} \subset Y$ is an affine algebraic set and the maps $X \longrightarrow \overline{\varphi(X)} \hookrightarrow Y$ are regular morphisms.

DOMINANT MORPHISMS, CLOSED EMBEDDINGS, AND FINITE MORPHISMS. If X is irreducible and the algebra homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$ is injective, then the corresponding morphism $X \xrightarrow{\varphi} Y$ is called *dominant*.

Geometrically, this means that $\overline{\varphi(X)} = Y$. If X is reducible, then φ is called *dominant* when its restriction onto *each* irreducible component of X is dominant. A morphism $X \xrightarrow{\varphi} Y$ is called *a closed embedding*, if $k[Y] \xrightarrow{\varphi^*} k[X]$ is surjective. This means that φ identifies X with $V(\ker\varphi^*) \subset Y$.

Exercice 11.19. Show that any dominant morphism of irreducible affine varieties $X \xrightarrow{\varphi} Y$ can be decomposed as

$$X \xhookrightarrow{\psi} Y \times \mathbb{A}^m \xrightarrow{\pi} Y \ , \tag{39}$$

where ψ is a closed embedding and π is the natural projection along $\mathbb{A}^m$.

HINT. Let $A = k[X]$, $B = k[Y]$. Since A is finitely generated B-algebra via the inclusion $B \xhookrightarrow{\varphi^*} A$, there is an epimorphism of B-algebras $B[x_1, x_2, \dots, x_m] \xrightarrow{\psi} A$ for some m.

For any regular morphism $X \xrightarrow{\varphi} Y$ the coordinate algebra $k[X]$ can be considered as an algebra over $\varphi^*(k[Y]) = k[\overline{\varphi(X)}] \subset k[X]$. A morphism φ is called *finite*, if $k[X]$ is integer over $\varphi^*(k[Y])$. Since $k[X]$ is finitely generated as algebra over $\varphi^*(k[Y])$ (even over k), finiteness of φ means that $k[X]$ is finitely generated also as *a $\varphi^*(k[Y])$-module*, i. e. there are some $f_1, f_2, \dots, f_m \in k[X]$ such that any $h \in k[X]$ can be written as $h = \sum \varphi^*(g_i)\, f_i$ for some $g_i \in k[Y]$.

Claim 11.20. *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of affine algebraic sets. Then $\varphi(Z)$ is closed for any closed $Z \subset X$ and induced morphism $Z \xrightarrow{\varphi|_Z} \varphi(Z)$ is finite. Moreover, if X is irreducible and $Z \neq X$, then $\varphi(Z) \neq Y$.*

Proof. Let $I = I(Z) \subset k[X]$ be the ideal of $Z \subset X$. Then $Z \xrightarrow{\varphi|_Z} Y$ has $\varphi_Z^* : k[Y] \xrightarrow{\varphi^*} k[X] \longrightarrow k[X]/I$. Since $k[X]$ is finitely generated as an $\varphi^*(k[Y])$-module, $k[Z] = k[X]/I$ is finitely generated as a module over $k[\overline{\varphi(Z)}] = \varphi|_Z^*(k[Y]) = \varphi^*(k[Y])/(I \cap \varphi^*(k[Y]))$, i. e. $Z \longrightarrow \overline{\varphi(Z)}$ is a finite morphism. To prove that $\varphi(Z) = \overline{\varphi(Z)}$, we can restrict ourself onto irreducible components of Z, i. e. suppose that Z is irreducible. Let $B = k[Z]$, $A = k[\overline{\varphi(Z)}] \subset B$, and $f_1, f_2, \dots, f_m$ generate B as A-module. Since $\varphi|_Z$ takes a maximal ideal $\mathfrak{m}_p \subset B$ to the maximal ideal $\mathfrak{m}_p \cap A \subset A$, a point $q \in Spec_m(A)$ does not belong to $Spec_m(B)$ iff its maximal ideal $\mathfrak{m}_q \subset A$ generates non proper ideal in B, that is $\mathfrak{m}_q \cdot B = B$. In this case we can write $f_i = \sum \beta_{i\nu} f_\nu$ for some $\beta_{i\nu} \in \mathfrak{m}_q$; that is the zero homomorphism of A-modules: $B \longrightarrow 0$, which takes each f_i to zero, can be presented in terms of the generator system $\{f_\nu\}$ by the matrix $E - (\beta_{\nu i})$. Hence, the multiplication by $\det(E - (\beta_{ij}))$ annihilates B. Since there no zero divisors in $B = k[Z]$, we get $\det(E - (\beta_{ij})) = 0$. But $\det(E - (\beta_{ij})) = 1 + \beta$ where $\beta \in \mathfrak{m}_q$. So, $1 \in \mathfrak{m}_q$ and $\mathfrak{m}_q \subset A$ is non proper as well.

To prove that $\varphi(Z) \neq Y$ for $Z \subsetneq X$, let us take a non zero function $f \in k[X]$ vanishing along Z and write the integer equation of the lowest possible degree for f over $\varphi^*(k[Y])$ as $f^m + \varphi^*(g_1)\, f^{m-1} + \dots + \varphi^*(g_{m-1})\, f + \varphi^*(g_m) = 0$. Computing its left side at any $z \in Z$, we get $\varphi^*(g_m)(z) = 0$, that is $g_m(\varphi(z)) = 0$. So, if $\varphi(Z) = Y$, then $g_m \equiv 0$ along Y, i. e. $\varphi^*(g_m) = 0$ in $k[X]$. Since $k[X]$ has no zero divisors, the minimal equation above is divisible by f. Contradiction. □

NORMAL ALGEBRAIC SETS. If Y is irreducible, then $k[Y]$ has no zero divisors and its quotient field is well defined; it is called *the field of rational functions* on Y and is denoted $k(Y)$. Irreducible Y is called *normal*, if its algebra $k[Y]$ is normal, i. e. if all rational functions $f \in k(Y)$ integer over $k[Y]$ belong to $k[Y]$. For example, affine spaces $\mathbb{A}^n$ are normal (as well as any X such that $k[X]$ is factorial).

Claim 11.21. *Let $X \xrightarrow{\varphi} Y$ be a surjective finite morphism. If Y is normal, then $\varphi(U)$ is open for any open $U \subset X$ (in particular each irreducible component of X is surjectively mapped onto Y).*

Proof. We can suppose that $U = \mathcal{D}(f)$ is principal. Let $p \in \mathcal{D}(f)$; we have to find $a \in k[Y]$ such that $\varphi(p) \in \mathcal{D}(a) \subset \varphi(\mathcal{D}(f))$. To this aim consider a map $\psi = \varphi \times f : X \xrightarrow{p \mapsto (\varphi(p), f(p))} Y \times \mathbb{A}^1$. It is regular: φ^* corresponds to an integer extension $k[Y] \subset k[X]$, i. e. $k[Y \times \mathbb{A}^1] = k[Y][t]$ and $k[Y][t] \xrightarrow{\psi^*} k[X]$ is the evaluation: $t \longmapsto f$. Since $k[X]$ is finitely generated even as $k[Y]$-module, ψ is finite. Now let us extend $k[X]$ to the $k(Y)$-algebra $B = k(Y) \underset{k[Y]}{\otimes} k[X]$, which consists of all fractions b/a, $b \in k[X]$, $0 \neq a \in k[Y]$ with the usual equivalence $b'/a' = b''/a'' \iff b''a' - b'a'' \in k[X]$ is a zero divisor. Since f is integer over $k[Y]$, it is algebraic over $k(Y)$, i. e. the kernel of evaluation map $k(Y)[t] \xrightarrow{t \mapsto f} B$ is the principal ideal $(\mu_f) \subset k(Y)[t]$ spanned by monic irreducible minimal polynomial $\mu_f(t) = t^m + a_1 t^{m-1} + \dots + a_{m-1}t + a_m \in k(Y)[t]$. Since Y is normal, Corollary 7.21 implies that μ_f has coefficients in $k[Y]$, i. e. $\mu_f \in k[Y \times \mathbb{A}^1]$. Hence, $\ker\psi^* = (\mu_f)$ and $\overline{\operatorname{im}\psi} = Spec_m\left(k[Y \times \mathbb{A}^1]/(\mu_f)\right) = V(\mu_f)$. In other words, ψ gives a finite surjection of X onto the hypersurface $V(\mu_f) \subset Y \times \mathbb{A}^1$ and φ is obtained by projecting this hypersurface onto Y. In particular, the preimages of given $y \in Y$ are surjectively mapped by f onto the roots of a polynomial $\mu_f(y;t) = t^m + a_1(y)\, t^{m-1} + \dots + a_m(y) \in k[t]$ obtained by evaluating the coefficients of μ_f at $y \in Y$. And those preimages which belong to $\mathcal{D}(f)$ correspond to non-zero roots. Since $f(p) \neq 0$, the polynomial $\mu_f(\varphi(p);t)$ has a non zero root. So, there is a non zero coefficient $a_i(\varphi(p)) \neq 0$. Then $\mu_f(y;t)$ has a non zero root over each $y \in \mathcal{D}(a_i) \subset Y$, i. e. $\varphi(\mathcal{D}(f)) \supset \mathcal{D}(a_i) \supset \varphi(p)$ as required. What about irreducible components, let $X = \cup X_\nu$ be the irreducible decomposition. Then $U_i = X \setminus \underset{\nu \neq i}{\cup} X_\nu = X_i \setminus \underset{\nu \neq i}{\cup} (X_i \cap X_\nu)$ is open in X and dense in X_i. Since $\varphi(U_i)$ is non empty open subset of Y, $\varphi(X_i) = \overline{\varphi(U_i)} = Y$. □