Sheaves.

PRE-SHEAVES. A concept of sheaf is one of the basic ground notions in mathematics just as a concept of set. By the definition *a pre-sheaf* F of groups (rings, modules, sets, ...) on a topological space X is *a contravariant functor* from a category of open sets (whose morphisms are inclusions) to the category of groups (rings, modules, sets, ...). In other words, F attaches, say a group, $F(U)$, called a group of *local sections*, to each open $U \subset X$ and attaches a group homomorphism $F(U) \xrightarrow{s \mapsto s|_W} F(W)$, called *a restriction* of local sections, to each pair $W \subset U$ in such a way that $\forall; W \subset U \subset V$ and any $s \in F(V)$ we have $(s|_U)|_W = s|_W$. By the definition, a morphism of pre-sheaves $E \longrightarrow F$ is a morphism of functors, that is a collection of homomorphisms $E(U) \longrightarrow F(U)$ defined for each U and commuting with the restrictions. Given $x \in X$, *a germ* of a local section of F at x is an equivalence class of pairs (U, s_U), where $U \ni x$ is open, $s_U \in F(U)$, and $(U, s_U) \sim (W, s_W)$, if $s_U|_V = s_W|_V$ for some open V such that $x \in V \subset U \cap W$. The germs of local sections at x form a group[1], which is called *a stalk* of F at x and denoted by F_x.

SHEAVES. A pre-sheaf F is called *a sheaf*, if for any two open sets U, W and any two sections $s_U \in F(U)$, $s_W \in F(W)$ such that $s_U|_{U\cap W} = s_W|_{U\cap W}$ there exists a unique $s \in F(U \cup W)$ such that $s|_U = s_U$ and $s|_U = s_U$. In particular, for any sheaf F

$$F(U \cup W) = F(U) \times F(W), \text{ if } U \cap W = \varnothing, \tag{43}$$

Exercice 15.1. Check that the structure sheaf of an algebraic variety (as it was defined in **lecture 11** and **lecture 12**) is really a sheaf of rings and the sheaf of local sections of an algebraic vector bundle is really a sheaf of modules over the structure sheaf.

For any pre-sheaf F there is a canonical sheaf $\widetilde{F}$ together with a morphism (of pre-sheaves) $F \xrightarrow{\sigma} \widetilde{F}$ such that for any sheaf E and any morphism (of pre-sheaves) $F \xrightarrow{\varphi} E$ there exists a unique morphism $\widetilde{F} \xrightarrow{\widetilde{\varphi}} E$ satisfying $\widetilde{\varphi} \circ \sigma = \varphi$. It is constructed as follows. For each open U consider the direct product of all stalks: $F_U \stackrel{\text{def}}{=} \prod_{x \in U} F_x$ and define $\widetilde{F}(U) \subset F_U$ as a subgroup which consists of all collections of germs $\{s_x\}_{x\in U}$ such that for any $y \in U$ there exist an open neighborhood $W \ni y$ and a local section $t \in F(W)$ whose germ (represented by a pair (W, t)) equals $s_x \ \forall\ x \in W \cap U$.

Exercice 15.2. Check that: (1) this prescription really defines a sheaf; (2) the universal property formulated before defines $\widetilde{F}$ up to a unique isomorphism commuting with σ.

Example 15.3: constant pre-sheaf and constant sheaf. Each, say a group, K trivially gives a pre-sheaf with the same section groups $K(U) = K \ \forall\ U$ and identical restrictions. This pre-sheaf is not a sheaf, because (43) falls. The associated sheaf $\widetilde{K}$ attaches to each U a group of *locally* constant, that is discrete but continuous, K-valued functions on U, which are constant only over *connected* U. It is called *a constant sheaf* of K.

Exercice 15.4. Compute the real primitive function $\int \frac{dx}{x}$.

HINT. Typical answer: «$\log|x| + C$, where C is a constant» is false: in fact, C is a section of the constant sheaf over $\mathbb{R} \setminus \{0\}$.

Example 15.5: factorization. Consider the unit circle S^1 and write $B \subset Z$ for the sheaves of local smooth closed and, respectively, exact real 1-forms on S^1. Then, globally, $Z(S^1)/B(S^1) = H^1_{\text{DR}}(S^1, \mathbb{R}) = \mathbb{R}$ is the first DeRahm cohomology group. It is spanned by global exact 1-form dt, where t is the natural length parameter. At the same time, locally, $Z(U) = B(U)$ over any connected open $U \subset S^1$. Hence, the pre-sheaf $U \longmapsto Z(U)/B(U)$ is not a sheaf: if connected open U, W cover the whole of S^1, then then zero and dt are two different global sections of Z/B which restricts to zero onto the both U, W.

Given two sheaves $F \subset E$ of Abelian groups on any topological space, then their factor sheaf E/F *is defined* as a sheaf associated with the pre-sheaf $U \longmapsto F(U)/E(U)$. Then, a section of E/F over U is a collection of germs (s_α, W_α) such that $s_\alpha \in F(W_\alpha)$, $W_\alpha \subset U$, and $(s_\alpha - s_\beta)|_{W_\alpha \cap W_\beta} \in E(W_\alpha \cap W_\beta)$. For example, a local section of the *factor sheaf* Z/B on S^1 over open $U \subset S^1$ is represented by a covering $U = \cup W_\alpha$ and a collection of closed forms $\vartheta_\alpha \in Z(W_\alpha)$ such that $(\vartheta_\alpha - \vartheta_b)|_{W_\alpha \cap W_\beta} \in B(W_\alpha \cap W_\beta)$.

Example 15.6: direct image (push forward) of a sheaf E on X w. r. t. a continuous map $X \xrightarrow{f} Y$ is a sheaf $f_\bullet E$ on Y whose fiber over an open $U \subset Y$ is $E(f^{-1}(U))$. If f is a closed embedding, then the sheaves $f_\bullet E$ on Y are called *sky scrapers* supported on X.

[1] or ring, module, set, ... — i. e. inherit the algebraic structure of the original local sections

Example 15.7: inverse image (pull back) of a sheaf F on Y w. r. t. a continuous map $X \xrightarrow{f} Y$ is a sheaf $f^\cdot F$ on X whose section over an open $U \subset X$ is a collection of germs $w_x \in F_{f(x)}$ (parameterized by $x \in U$) such that for any $y \in U$ there exist an open $W \ni f(y)$ in Y and a section $s \in F(W)$ whose germ (represented by (s, W)) coincides with w_x for all x from some open neighborhood $V \subset f^{-1}(W) \cap U$, $V \ni y$. In particular, a constant sheaf on X associated with K is a pull back of K under the contraction of X to one point and a stalk of a sheaf at a point is a pull back of this sheaf under the inclusion of the point into the manifold.

Exercice 15.8. Check that $f^\cdot F$ is really a sheaf on X even for a pre-sheaf F on Y (moreover, a sheaf associated with a given pre-sheaf is nothing but a its pull back under the identity map) and prove that for any sheaves E on X and F on Y there exists a natural isomorphism $\mathrm{Hom}_X(f^\cdot F, E) = \mathrm{Hom}_Y(F, f_\cdot E)$.

Example 15.9: localization. Let X be an algebraic manifold (or geometric scheme) with the structure sheaf $\mathscr{O}_X$ and $Z \xhookrightarrow{f} X$ be a closed embedding. Then $f^\cdot \mathscr{O}_X$ is called *a localization* of regular functions on X near Z. For example, if X is affine, $\mathscr{O}_X(X) = A$, and $Z \subset X$ is an irreducible submanifold given by a prime ideal $\mathfrak{p} \subset A$, then $f^\cdot \mathscr{O}_X(Z)$ is a ring of fractions[2] $A_\mathfrak{p} = \{p/q \mid p, q \in A,\ q \notin \mathfrak{p}\}$ (these fractions are added and multiplied by the usual rules and, by the definition, $p_1/q_1 = p_2/q_2$ iff $q_2 p_1 - q_1 p_2$ is a zero divisor in A). This is a *local ring*, i. e. it contains a unique proper maximal ideal $\mathfrak{m}_\mathfrak{p} = \{p/q \mid p \in \mathfrak{p},\ q \notin \mathfrak{p}\}$ and all elements of $A_\mathfrak{p} \setminus \mathfrak{m}_\mathfrak{p}$ are invertible. In particular, if $Z = x$ is one point with maximal ideal $\mathfrak{m}_x$, then we get the local ring A_x, which coincides with the stalk $\mathscr{O}_{X\,x}$ and consists of germs of local rational functions well defined at x. Its unique maximal ideal $\mathfrak{m}_x \subset A_x$ consists of all rational functions vanishing at x; so $A_x/\mathfrak{m}_x = k$.

$\mathscr{O}$-MODULES. In the framework of algebraic geometry, the sheaves of $\mathscr{O}_X$-modules are those usually considered on an algebraic manifold X. Sections of such a sheaf over an open $U \subset X$ form a module over $\mathscr{O}_X(U)$. A regular map of algebraic manifolds (or geometric schemes) $X \xrightarrow{f} Y$ involves the homomorphism of structure sheaves[3] $\mathscr{O}_Y \xrightarrow{f^*} f_\cdot \mathscr{O}_X$, or equivalently (see the isomorphism from Exercice 15.8), the homomorphism $f^\cdot \mathscr{O}_Y \xrightarrow{f^*} \mathscr{O}_X$. The both send a local regular function on Y to its composition with f and endow $f_\cdot \mathscr{O}_X$ with the structure of $\mathscr{O}_Y$-algebra and $\mathscr{O}_X$ — with the structure of $f^\cdot \mathscr{O}_Y$-algebra. So, given a sheaf E of $\mathscr{O}_X$-modules on X, then, restricting scalars from $f_\cdot \mathscr{O}_X$ to $\mathscr{O}_Y$, we can consider the $f_\cdot \mathscr{O}_X$-module $f_\cdot E$ as an $\mathscr{O}_Y$-module as well. This sheaf of $\mathscr{O}_Y$-modules on Y is denoted by $f_* E$ and is called a *push forward* of the $\mathscr{O}_X$-module E. Dually, given an $\mathscr{O}_Y$-module F on Y, then we can extend scalars from $f^\cdot \mathscr{O}_Y$ to $\mathscr{O}_X$ in the $f^\cdot \mathscr{O}_Y$-module $f^\cdot F$. The resulting $\mathscr{O}_X$-module $\mathscr{O}_X \underset{f^\cdot \mathscr{O}_Y}{\otimes} f^\cdot F$ on X is called a pull back of the $\mathscr{O}_Y$-module F on Y and is denoted by $f^* F$.

QUASI-COHERENT SHEAVES. A sheaf F of $\mathscr{O}_X$-modules over an affine algebraic manifold X is called *quasi-coherent*, if it is constructed from some $k[X]$-module M_F in the same way as the sheaf of sections of locally trivial vector bundle E was recovered from the module $P_E = \Gamma(X, E)$, of its global sections, in Claim 14.12. More precisely, quasi-coherent sheaf $\widetilde{M}$ associated with $k[X]$-module M takes[4] $\widetilde{M}(U) = \mathscr{O}_X(U) \underset{k[X]}{\otimes} M$ over each open $U \subset X$. In other words, a section of $\widetilde{M}$ over U is a collection of fractions m_x/f_x parameterized by $x \in U$ such that $\forall\, y \in U$ there exist an open neighborhood $V \subset U$, $V \ni y$ and a pair $m \in M$, $f \in \mathscr{O}_X(V)$ satisfying two properties: $f(y) \neq 0$ and $m_x/f_x = m/f \quad \forall\, x \in V \cap \mathcal{D}(f)$. A quasi coherent sheaf $\widetilde{M}$ is called *coherent*, if the module M is finitely generated over $k[X]$. In general case, an $\mathscr{O}_X$-module $\mathscr{M}$ over an arbitrary geometric scheme X is called quasicoherent (resp. coherent), if there exists an affine open covering (resp. a finite affine open covering)$X = \cup X_i$ such that $\mathscr{M}(X_i)$ is a quasicoherent (resp. coherent) $\mathscr{O}_{X_i}$-module for each i. The quasi coherence is preserved by pull-backs and push forwards, moreover, the pull-back of a coherent module is also coherent. In a contrast, the direct image of a coherent module is often non finitely generated: for example, contracting X to a point p, we restrict scalars from $k[X]$ to $k\mathscr{O}_p$ when take a push forward and $\mathscr{O}_X$, say, becomes infinite dimensional; an other typical example gives the inclusion $X = \mathbb{A}_1 \setminus \{0\} \xhookrightarrow{i} \mathbb{A}_1$ where $i_*\mathscr{O}_X = \widetilde{k[t, t^{-1}]}$, which is not finitely generated over $\widetilde{k[t]} = \mathscr{O}_{\mathbb{A}_1}$.

Example 15.10: sheaves of ideals and closed subschemes. $\mathscr{O}_X$-submodule $\mathscr{I} \subset \mathscr{O}_X$ is called *a sheaf of ideals*. It is coherent, because its sections $\mathscr{I}(U)$ over affine $U \subset X$ form (maybe non reducible) ideal $I \subset k[U]$. The zero set $Z_I \subset U$ is a closed algebraic subset of U equipped with the structure of geometric scheme given by the k-algebra $k[U]/I$. These local closed subschemes are glued together into one closed geometric subscheme Z equipped with the regular map $Z \xhookrightarrow{f} X$ whose pull back takes $\mathscr{O}_X$-module E to $\mathscr{O}_Z = \mathscr{O}_X/\mathscr{I}$ module $\left(\mathscr{O}_X/\mathscr{I}\right) \underset{\mathscr{O}_X}{\otimes} E$. Sines multiplication by $\mathscr{I}$ acts trivially

[2] also called a *localization* of A at $\mathfrak{p}$

[3] it is uniquely defined by f in the case of manifolds and is a part of definition of a regular map in the case of geometric schemes

[4] the tensor product $* \underset{k[X]}{\otimes} *$ used here is the tensor product of $k[X]$-modules defined precisely as in Exercice 14.1: if K is a k-algebra and M_1, M_2 — two K-modules, then $M_1 \underset{K}{\otimes} M_2$ is the quotient of $M_1 \otimes M_2$ by a subspace spanned by all $(\varkappa m_1) \otimes m_2 - m_1 \otimes (\varkappa m_2)$, where $\varkappa \in K$, $m_i \in M_i$

on this module, all its sections over any open $W \subset X \setminus Z$, where $\mathscr{I}(W) = \mathscr{O}_X(W)$, should vanish. So, Z can be also described as the support of the factor sheaf $F = \mathscr{O}_X/\mathscr{I}$, i. e. $Z = \{x \in X \mid F_x \neq 0\}$. By this reason, a pull back map along closed embedding $Z \hookrightarrow X$ is usually called a *restriction* of an $\mathscr{O}_X$-modules onto Z. In particular, if $Z = x$ is one point and $\mathscr{I} = \mathfrak{m}_x$ is its maximal ideal, then the restriction of an $\mathscr{O}_X$-module E onto x, i. e. $(\mathscr{O}_X/\mathfrak{m}_x) \underset{\mathscr{O}_X}{\otimes} E$, is nothing but $E_x/\mathfrak{m}_x \cdot E_x$, where E_x is a stalk of E at x. The ideal sheaf $\mathscr{I}$ of a closed subscheme $Z \subset X$ is locally free iff Z can be given locally by one equation; in particular, an ideal sheaf of any nonempty subscheme of codimension $\geqslant 2$ is not locally free.

SHEAVES OF ALGEBRAS AND AFFINE MORPHISMS. Given a sheaf $\mathscr{A}$ of $\mathscr{O}_X$-algebras which is quasi coherent as a sheaf of $\mathscr{O}_X$-modules, then over any affine $U \subset X$ we have a finitely generated $k[U]$-algebra A and two maps: $k[U] \xrightarrow{\alpha^*} A$ and $k[U][x_1, x_2, \dots, x_n] \twoheadrightarrow A$ which give a map $Spec_m A \xrightarrow{\alpha} U$ and a closed embedding $Spec_m A \hookrightarrow U \times \mathbb{A}_n$. The maps α are glued together into a global map $Y \xrightarrow{\widetilde{a}} X$, where Y is a geometric scheme with a structure sheaf $\mathscr{A}$ locally (over affine $U \subset X$) isomorphic to a closed subscheme in $U \times \mathbb{A}_n$ (in general, n depends on U). Morphisms of the type $Y \longrightarrow X$ just described are called *affine morphisms*. The Example 15.10 can be considered as a particular case of this construction where $\mathscr{A} = \mathscr{O}_X/\mathscr{I}$. An other important particular case is given in the next example.

Example 15.11: vector bundles. Given a coherent $\mathscr{O}_X$-module F, a sheaf of its symmetric algebras[5]

$$\mathcal{S}(F) = \bigoplus_{d \geqslant 0} S^d(F) \tag{44}$$

satisfies the above conditions, i. e. is a quasi coherent $\mathscr{O}_X$-algebra. The corresponding affine morphism is denoted as $\mathbb{V}(F^*) \xrightarrow{\pi} X$ and called a *vector bundle dual to* F. By the construction, the fiber $\pi^{-1}(p)$ over a given $p \in X$ is the maximal spectrum of the k-algebra $(\mathscr{O}_X/\mathfrak{m}_x) \underset{\mathscr{O}_X}{\otimes} \mathcal{S}(F) = \bigoplus_{d \geqslant 0} S^d(F_x/\mathfrak{m}_x \cdot F_x)$ (comp. with the Example 15.10), that is a vector space $(F_x/\mathfrak{m}_x \cdot F_x)^*$ over $k = \mathscr{O}_{X\,x}/(\mathfrak{m}_x \cdot \mathscr{O}_{X\,x})$ (recall that $\mathbb{A}(V) = Spec_m\,(\bigoplus_{d \geqslant 0} S^d V^*)$). In other words, $\mathbb{V}(F) \xrightarrow{\pi} X$ is a vector bundle (possibly not locally trivial) and sections of F are naturally identified with functions $\mathbb{V}(F) \longrightarrow k$ linear along fibers of π.

Exercice 15.12. Show that vector bundle $\mathbb{V}(F)$ is locally trivial iff F is locally free $\mathscr{O}_X$-module.
HINT. See Corollary 14.13.

SHEAVES OF GRADED ALGEBRAS, PROJECTIVE MORPHISMS, AND CONES. Given a sheaf $\mathscr{A} = \bigoplus_{m \geqslant 0} \mathscr{A}_m$ of *graded* $\mathscr{O}_X$-algebras such that: (1) $\mathscr{A}_m = \mathscr{O}_X$; (2) $\mathscr{A}_1$ is coherent $\mathscr{O}_X$-module; (3) $\mathscr{A}$ is generated by $\mathscr{A}_1$ as $\mathscr{O}_X$-algebra, then $\mathscr{A}$ is quasicoherent and over any affine $U \subset X$ we have a finitely generated graded $k[U]$-algebra $A = \oplus A_k = k[U]\langle x_1, x_2, \dots, x_n\rangle / I$, where $x_0, x_1, \dots, x_n \in A_1$ are some generators of $k[U]$-module A_1 (in general, n may depend on U) and $I \subset k[U]\langle x_1, x_2, \dots, x_n\rangle$ is a graded ideal. So, we have a closed subscheme $Y_U \subset U \times \mathbb{P}_n$ given by I and equipped with the closed morphism $Y_U \xrightarrow{p_U} U$ induced by the projection $U \times \mathbb{P}_n \twoheadrightarrow U$. The maps p_U are glued together into a global map $Y \xrightarrow{p} X$. Such a map p is called a *projective morphism* and the geometric scheme Y is denoted $Proj(\mathscr{A})$ and called a *projective spectrum* of graded $\mathscr{O}_X$-algebra $\mathscr{A}$. Note that epimorphism of $\mathscr{O}_X$-algebras $\mathcal{S}(\mathscr{A}_1) \xrightarrow{\varphi^*} \mathscr{A}$ (induced by the multiplication) gives a closed embedding $Spec_m \mathscr{A} \xrightarrow{\varphi} \mathbb{V}(\mathscr{A}_1^*)$ and $Proj(\mathscr{A}) \subset \mathbb{PV}(\mathscr{A}_1^*)$ is nothing but the fiberwise projectivization of $\operatorname{im}\varphi$ (see Example 15.13 below).

Example 15.13: projective bundles. As in Example 15.11, the symmetric algebra (44) of any coherent $\mathscr{O}_X$-module F satisfies the above conditions (1) – (3), that is produces the projective morphism $Proj(\mathcal{S}(F)) \xrightarrow{p} X$. Certainly, the manifold $Proj(\mathcal{S}(F))$ is nothing but the projective bundle obtained from the vector bundle $\mathbb{V}(F^*)$ (see Example 15.11) by the fiberwise projectivization. It usually denoted by $\mathbb{P}(F^*)$ or $\mathbb{P}(\mathbb{V}(F^*))$, because the fiber $p^{-1}(x)$ at $x \in X$ is the projective space $\mathbb{P}\,((F_x/\mathfrak{m}_x \cdot F_x)^*)$ over $k = \mathscr{O}_{X\,x}/(\mathfrak{m}_x \cdot \mathscr{O}_{X\,x})$ and sections of F can be considered as fiberwise linear homogeneous forms on $\mathbb{P}(F^*)$. Such the forms form a locally free sheaf $\mathscr{O}_{\mathbb{P}(F^*)}$-modules of rank 1; this sheaf (as well as the corresponding line bundle over $\mathbb{P}(\mathbb{V}(F^*))$) is denoted by $\mathscr{O}_{\mathbb{P}(F^*)}(1)$.

In general case, the restriction of the line bundle $\mathscr{O}_{\mathbb{PV}(\mathscr{A}_1^*)}(1)$ from $\mathbb{PV}(\mathscr{A}_1^*)$ onto $Proj(\mathscr{A})$ is denoted $\mathscr{O}_{Proj(\mathscr{A})}(1)$ as well.

Non projectivized affine morphism $Spec_m \mathscr{A} \xrightarrow{\pi} X$ is called a *dual cone* of the graded $\mathscr{O}_X$-algebra $\mathscr{A}$ and denoted $C(\mathscr{A})$. There is canonical *zero section* $X \xrightarrow{\sigma_0} C(\mathscr{A})$ given by the augumentation homomorphism $\mathscr{A} \xrightarrow{\sigma_0^*} \mathscr{O}_X$, which is identical on $\mathscr{A}_0$ and takes to zero all $\mathscr{A}_i$ with $i > 0$.

[5] the symmetric algebra of an $\mathscr{O}_X$-module is defined precisely in the same way as the symmetric algebra of a vector space (see Claim 4.13 and discussion around it) with only difference: all the tensor products are taken over $\mathscr{O}_X$ rather than over k