Projective spaces, affine charts, coordinates, and projections

(Oct. 15 — Oct. 29)

1◇1. Let S^dV^* be the space of all degree d homogeneous polynomials on n-dimensional linear space V. Find $\dim S^dV^*$.

1◇2 (Veronese map). Under the previous problem conditions, let $V^* \xrightarrow{v_d} S^dV^*$ take a linear form ℓ to its d-th power ℓ^d. Does the image of v_d lay in a hyperplane or its linear span exhaust the whole of S^dV^*?

1◇3. Let V be n-dimensional vector space over a finite field consisting of q elements. How many **a)** basises **b)** k-dimensional subspaces are there in V? **c)** How many points are there in $\mathbb{P}(V)$?

1◇4*. Let $G_n^k(q)$ be the number of k-dimensional vector subspaces in n-dimensional vector space over a finite field of q elements. Compute $\lim\limits_{q\to 1} G_n^k(q)$.

1◇5. Consider the projective closures of affine curves
a) $y = x^2$ **b)** $y = x^3$ **c)** $y^2 + (x-1)^2 = 1$ **d)** $y^2 = x^2(x+1)$
Write down their homogeneous equations and their affine equations in two other standard affine charts on $\mathbb{P}_2$. Try to draw all these affine curves.

1◇6. Let the real Euclidian plane $\mathbb{R}^2$ be included in $\mathbb{CP}_2$ as the real part of the standard affine chart U_0. Find two points of $\mathbb{CP}_2$ such that any Euclidean circle will contain them after comlexification and projective closuring.

1◇7 (Pythagorean triples). Consider $\mathbb{P}_2$ with homogeneous coordinates $(t_0 : t_1 : t_2)$. Let $\ell \subset \mathbb{P}_2$ be the line $t_2 = 0$, $Q \subset \mathbb{P}_2$ be the conic $t_0^2 + t_1^2 = t_2^2$, and $O = (1:0:1) \in Q$. For each $P = (p:q:0) \in \ell$ find coordinates of the intersection point $Q \cap (OP)$ different from O and show that the projection from O maps Q bijectively onto ℓ. Find some polynomials $a(p,q)$, $b(p,q)$, $c(p,q)$ whose values on $\mathbb{Z}\times\mathbb{Z}$ give, up to a common factor, all integer Pythagorian triples $a^2 + b^2 = c^2$ (and only such the triples).

1◇8 (projecting twisted cubic). Let $\mathbb{P}_1 = \mathbb{P}(V^*)$ be the space of linear forms (considered up to constant factor) in two variables (t_0, t_1), $\mathbb{P}_3 = \mathbb{P}(S^3V^*)$ be the space of cubic forms in (t_0, t_1), and $\mathbb{P}_1 \xhookrightarrow{v_3} \mathbb{P}_3$ be the cubic Veronese map, like in prb. 1◇2. Its image is called *the twisted cubic* C_3. Calculate the projection of C_3:

a) from the point t_0^3 to the plane spanned by $3\,t_0^2t_1$, $3\,t_0t_1^2$, and t_1^3
b) from the point $3\,t_0^2t_1$ to the plane spanned by t_0^3, $3\,t_0t_1^2$, and t_1^3
c) from the point $t_0^3 + t_1^3$ to the plane spanned by t_0^3, $3\,t_0^2t_1$, and $3\,t_0t_1^2$

More precisely, write explicit parametric representation for the projection in appropriate coordinates, then find its affine and homogeneous equation. Do that for several affine charts on the projection target plane. In each case, find degree of the the curve and try to draw it. Has it selfintersections and/or cusps?