Quadrics and conics[1]

(Oct. 22 — Nov. 5)

2⋄1. Consider the quadratic form $q(A) = \det A$ on the space of square 2×2-matrices $Mat_2(k)$. Describe its polarization, i. e. what is the number $\widetilde{q}(A,B)$ in terms of two matrices[2] A and B.

2⋄2 (continuation of prb. 1⋄6). Under the conditions of prb. 1⋄6, show that any quadric on $\mathbb{CP}_2$, which pass through two points you have found in prb. 1⋄6 and has at least 3 points inside the initial real Euclidian plane, looks there as a circle.

2⋄3 (Euclidean polarities). Consider a circle in the real Euclidean affine plane. How to draw:

a) the polar of a given point (especially, when the point is inside the circle)

b) the pole of a given line (especially, when the line does not intersect the circle)

Describe geometrically a polarity w. r. t. an «imaginary circle» $x^2 + y^2 = -1$.

2⋄4. Show that all conics passing through the points $a = (1 : 0 : 0)$, $b = (0 : 1 : 0)$, $c = (0 : 0 : 1)$, $d = (1 : 1 : 1)$ form a line in the space of all conics. Write an explicit equation[3] for these conic family and find all singular conics inside it.

2⋄5 (1 – 1 correspondence on a conic). Let $Q \subset \mathbb{P}_2$ be non singular conic considered together with some rational parameterization $\mathbb{P}_1 \xrightarrow{\sim} Q$. Show that for any bijection $Q \xrightarrow{\gamma} Q$ induced by a linear automorphism of $\mathbb{P}_1$ there exist two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ such that $x \overset{\gamma}{\longmapsto} y$ iff $\pi_\ell^{p_1}\, x = \pi_\ell^{p_2}\, y$. Were are the fixed points of this map? Is it possible, using only the ruler, to find p_1, p_2, ℓ for γ given by its action on 3 points $a, b, c, \in Q$?

2⋄6*. Using only the ruler, draw a triangle inscribed in a given non singular conic Q and such that his sides a, b, c pass through 3 given points A, B, C. How many solutions may have this problem?

HINT. Start «naive» drawing from any $p \in Q$ and denote by $\gamma(p)$ your return point after passing trough A, B, C. Is $p \longmapsto \gamma(p)$ a projective isomorphism of kind described in prb. 2⋄5?

2⋄7*. Formulate and solve projectively dual problem to the previous one.

2⋄8*. Describe a general algorithm for reducing a trigonometric equation $f(\sin(x), \cos(x)) = 0$, where f is an arbitrary quadratic polynomial in two variables, to a simple equation $\cos(x) = \alpha$, where α contains at most cubic irrationalities.

HINT. The problem is how to compute explicitly 4 intersection points of 2 quadrics $f(x, y) = 0$ and $x^2 + y^2 = 1$; but the same intersection points can be produced by any two quadrics from the same pencil. A good idea is to intersect two singular conics of this pencil.

2⋄9. How many lines cross each of 4 given pairwise skew lines in:

a) $\mathbb{CP}_3$ **b)** $\mathbb{RP}_3$ **c*)** $\mathbb{C}^3$ **d*)** $\mathbb{R}^3$

Find all possible answers and indicate those are stable under small perturbations.

2⋄10. Consider the Segre embedding[4] $\mathbb{P}(U_-^*) \times \mathbb{P}(U_+) \xrightarrow{\sim} Q_S \subset \mathbb{P}(\mathrm{Hom}(U_-, U_+))$ and let a point $p \in \mathbb{P}(\mathrm{Hom}(U_-, U_+)) \setminus Q_S$ correspond to an isomorphism $\mathbb{P}(U_-) \xrightarrow{\varphi} \mathbb{P}(U_+)$. Consider any $u \in \mathbb{P}(U_-)$ and let $\xi = \mathrm{Ann}\,(u) \in \mathbb{P}(U_-^*)$ ([5]). Draw a plane $\pi_{p,\ell}$ through the point p and the line $\ell = s(\xi \times \mathbb{P}_1) \subset Q_S$. Prove that $\pi_{p,\ell} \cap Q_S$ is a reducible conic of the form $\ell \cup \ell'$, where the second component ℓ' has the form $s(\mathbb{P}_1 \times t)$ for some $t \in \mathbb{P}(U_+)$, which coincides with $\varphi(u)$.

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[1] problems marked with "star" $*$ are rather honorary than obligatory

[2] for example, the standard Euclidean norm $\|\,(a_{ij})\,\| \overset{\text{def}}{=} \sum a_{ij}^2$ is polarized to $(A, B) = \mathrm{tr}\ (A \cdot {}^t\!B)$; polarization of $\det(A)$ should look similarly with something else instead of ${}^t\!B$ ...

[3] it should be a quadratic form whose coefficients depend linearly on two homogeneous parameters

[4] see lecture 4 and "Tensor Guide"

[5] i. e. if $\xi = (\xi_0 : \xi_1)$, then $u = (\xi_1 : -\xi_0)$