Plücker – Segre – Veronese interaction

(Oct. 29 — Nov. 12)

3◇1. Consider two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ and denote by $\ell_1^\times, \ell_2^\times \subset \mathbb{P}_3^\times$ two pencils of planes passing through these lines. Take any 3 non collinear points a, b, c such that no two of them are coplanar with either ℓ_1 or ℓ_2. Write $\ell_1^\times \xrightarrow{\gamma_{abc}} \ell_2^\times$ for a linear projective isomorphism that sends 3 planes passing through a, b, c in $\ell_1^\times$ to the corresponding planes in $\ell_2^\times$.

Describe the incidence graph $\Gamma_{abc} \overset{\text{def}}{=} \bigcup\limits_{\pi \in \ell_1^\times} \big(\pi \cap \gamma_{abc}(\pi)\big)$ ruled by the intersection lines of γ_{abc}-incident planes, if: **a)** $\ell_1 \cap \ell_2 = \varnothing$ **b)** $\ell_1 \cap \ell_2 \neq \varnothing$

3◇2. Let $A \in \operatorname{Hom}(U, V) \simeq U^* \otimes V$, $B \in \operatorname{Hom}(V, W) \simeq V^* \otimes W$ be two linear maps decomposed as $A = \sum \alpha_\nu \otimes a_\nu$, $B = \sum \beta_\mu \otimes b_\mu$ with $\alpha_\nu \in U^*$, $a_\nu \in V$, $\beta_\mu \in V^*$, $b_\mu \in W$. Decompose similarly their product $B \circ A \in \operatorname{Hom}(U, W) \simeq U^* \otimes W$.

3◇3. Are the following decompositions valid for any vector space V over a field of zero characteristic: **a)** $V^{\otimes 2} \simeq S^2V \oplus \Lambda^2 V$ **b)** $V^{\otimes 3} \simeq S^3V \oplus \Lambda^3 V$ (If yes, give a proof, if no, give an explicit example of a tensor, which can not be decomposed in this way.)

3◇4 (spinor decomposition). Let $V = \operatorname{Hom}(U_-, U_+)$, where $\dim U_\pm = 2$. Show that $V = U_-^* \otimes U_+$ and $V^{\otimes 2} = \underbrace{\big((S^2U_-^* \otimes S^2U_+) \oplus (\Lambda^2U_-^* \otimes \Lambda^2U_+) \big)}_{S^2V} \bigoplus \underbrace{\big((S^2U_-^* \otimes \Lambda^2U_+) \oplus (\Lambda^2U_-^* \otimes S^2U_+) \big)}_{\Lambda^2V}$.

3◇5. Let $G \subset \mathbb{P}_3 = \mathbb{P}(V)$ be a non singular quadric given by a quadratic form g whose polarization is $\widetilde{g}$. Show that bilinear form $\Lambda^2\widetilde{g}$ on Λ^2V, which acts on decomposable bivectors as

$$\Lambda^2\widetilde{g}(\, v_1 \wedge v_2 \,,\, w_1 \wedge w_2 \,) \overset{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1, w_1) & \widetilde{g}(v_1, w_2) \\ \widetilde{g}(v_2, w_1) & \widetilde{g}(v_2, w_2) \end{pmatrix} ,$$

is symmetric and non degenerate, and write its explicit Gram matrix in a convenient base (say, coming from the orthonormal base of g in V). Show that the intersection of the corresponding quadric $\Lambda^2 G \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ with the Plücker quadric consists of all tangent lines to $G \subset \mathbb{P}_3$.

3◇6. Under the previous problems notations, let $\operatorname{Gr}(2, V)$ be the Grassmannian variety, of lines in $\mathbb{P}_3 = \mathbb{P}(V)$. Show that the Plücker embedding $\operatorname{Gr}(2, V) \hookrightarrow \mathbb{P}(\Lambda^2 V)$ sends two line families living on the Segre quadric $G \subset \mathbb{P}(V) = \mathbb{P}(\operatorname{Hom}(U_-, U_+))$ to a pair of non singular plane conics that are cut out the Plücker quadric $P \subset \mathbb{P}(\Lambda^2 V)$ by two complementary planes $\Lambda_- = \mathbb{P}\left(S^2U_-^* \otimes \Lambda^2 U_+\right)$ and $\Lambda_+ = \mathbb{P}\left(\Lambda^2 U_-^* \otimes S^2 U_+\right)$ laying in $\mathbb{P}(\Lambda^2 \operatorname{Hom}(U_-, U_+))$ via prb. 3◇4. Moreover, the both conics are embedded into these planes via Veronese, that is, we have the following commutative diagram (Plücker is dotted, because it maps lines into points):

$$\begin{tikzcd}
\mathbb{P}(U_+) \arrow[rr, hook, "\text{Veronese}"] & & \mathbb{P}(S^2U_+) \simeq \Lambda_+ \arrow[d, hook] \\
\mathbb{P}_1^+ \times \mathbb{P}_1^- \arrow[u, "\pi_+"] \arrow[d, "\pi_-"'] \arrow[r, "\text{Segre}", "\sim"'] & G \subset \mathbb{P}\operatorname{Hom}(U_-, U_+) \arrow[r, dashed, "\text{Plücker}"] & P \subset \mathbb{P}\begin{pmatrix} \Lambda^2U_-^* \otimes S^2U_+ \\ \oplus \\ S^2U_-^* \otimes \Lambda^2 U_+ \end{pmatrix} \\
\mathbb{P}(U_-^*) \arrow[rr, hook, "\text{Veronese}"'] & & \mathbb{P}(S^2U_-^*) \simeq \Lambda_- \arrow[u, hook]
\end{tikzcd}$$

3◇7. Is there a 2×4 - matrix whose 2×2 - minors are: **a)** $\{2, 3, 4, 5, 6, 7\}$ **b)** $\{3, 4, 5, 6, 7, 8\}$ (If no, explain why, if yes, give an explicit example.)