Some useful (multi) linear algebra

(Nov. 5 — Nov. 19)

4◇1. Is it true that any rank 1 matrix of size $m \times n$ can be written as a product of some $m \times 1$ and $1 \times n$ matrices?

4◇2. Show that for any two vector spaces U, W there exists a canonical isomorphism

$$U^* \otimes W \xrightarrow{\sim} \mathrm{Hom}(U, W)\ ,$$

which sends $\xi \otimes w$ to rank 1 operator $u \longmapsto \xi(u) \cdot w$.

4◇3. Let $\{e_1, e_2, \dots, e_d\} \subset V$ and $\{x_1, x_2, \dots, x_d\} \subset V^*$ be dual bases. Show that the tensor $\sum\limits_\nu x_\nu \otimes e_\nu \in V^* \otimes V$ doesn't depend on a choice of the dual bases.

4◇4. Check for any vector space V a series of canonical isomorphisms:

$$\mathrm{Hom}(V, V) \simeq V^* \otimes V \xrightarrow{\tau} (V \otimes V^*)^* \simeq \mathrm{Hom}(V, V)^*$$

where τ takes $\xi \otimes v$ to a linear form that sends $v' \otimes \xi'$ to the full contraction $\xi(v')\xi'(v)$. The resulting correlation $\mathrm{Hom}(V, V) \xrightarrow{\sim} \mathrm{Hom}(V, V)^*$ corresponds to some bilinear form $t(A, B) \overset{\text{def}}{=} \tau A(B)$ on $\mathrm{Hom}(V, V)$. Is this form symmetric? How it looks in terms of matrices? What is the corresponding quadratic form?

4◇5. Let $A = (a_{ij})$ be $n \times n$ - matrix whose entries are considered as independent variables. Fix a collection of m matrix elements $a_{i_\nu j_\nu}$, where $1 \leqslant \alpha \leqslant m$. Calculate the polarization

$$\frac{\partial^m \det A}{\partial a_{i_1 j_1}\, \partial a_{i_2 j_2} \cdots \partial a_{i_m j_m}}$$

a) for $m = 1$; **b)** for $m = 2$; **c*)** for any m.

Hint. Use Laplace relations: write A_m for $\binom{n}{m} \times \binom{n}{m}$ matrix whose entries are $m \times m$-minors of A and $\widehat{A}_m$ for the matrix of their algebraic complements; then $\det A = \binom{n}{m}^{-1} \mathrm{tr}\left(A_m \cdot {}^t\!\widehat{A}_m\right)$.

4◇6*. Prove the following Taylor expansion for the polynomial function $\det(A)$ on the space of $n \times n$ - matrices:

$$\det(\lambda A + \mu B) = \sum_{p+q=n} \binom{n}{p}^{-1} \cdot \lambda^p \mu^q \cdot \sum_{\substack{IJ:\\ \#I=\#J=p}} (-1)^{|I|+|J|} a_{IJ} b_{\widehat{I}\widehat{J}} \tag{1}$$

where (a_{IJ}) is $p \times p$-minor of $A = (a_{ij})$ situated in rows $I = (i_1, i_2, \dots, i_p)$ and columns $J = (j_1, j_2, \dots, j_p)$ and $\left(b_{\widehat{I}\widehat{J}}\right)$ is $q \times q$-minor of $B = (b_{ij})$ situated in the complementary rows $\widehat{I} = \{1, 2, \dots, n\} \setminus I$ and the complementary columns $\widehat{J} = \{1, 2, \dots, n\} \setminus J$ (in the notations from the hint to the previous problem, the last right sum in (1) actually equals tr $\left(A_p \cdot \widehat{B}_q\right)$).

4◇7. Let $e = \{e_1, e_2, \dots, e_d\} \subset V$ be a base. Denote by $\sigma_\mu(e) \subset \mathrm{Gr}(n, V)$ a Schubert cycle defined as the closure of all U whose matrix with respect to the base e has a strong step form predicted by the Young diagram μ. Show that if two Schubert cycles $\sigma_\lambda(e)$ and $\sigma_\mu(e')$ have complementary non zero codimensions $|\lambda| + |\mu| = n(d-n)$ and two bases e and e' are general enough, then $\sigma_\lambda(e) \cap \sigma_\mu(e') = \varnothing$ for all λ, μ except for the case when the diagrams λ, μ are complementary, i. e. $\mu_{n+1-i} = d-n+1-\lambda_i$ (if we put them together appropriately, then we get the $n \times (d-n)$-rectangle). Show also that for complementary diagrams the intersection consists, typically, of one point.

Hint. To start with, take $e'_i = e_{d+1-i}$