Hypersurfaces, tangents, and singularities[1]

(Nov. 12 — Nov. 26)

5◇1. In $\mathbb{P}_n$, consider 2-dimensional plane π and two codimension 2 subspaces L_1, L_2 such that $p_1 = L_1 \cap \pi$ and $p_2 = L_2 \cap \pi$ are two distinct points on π. Take $a, b, c \in \pi$ such that no 3 of p_1, p_2, a, b, c are collinear. Write $\ell_1 = L_1^\times \subset \mathbb{P}_n^\times$, $\ell_2 = L_2^\times \subset \mathbb{P}_n^\times$ for two pencils of hyperplanes passing through L_1, L_2 respectively, and $\gamma_{abc} \colon \ell_1 \xrightarrow{\sim} \ell_2$ — for the usual projective linear isomorphism defined by a, b, c. Show that its incidence graph $\bigcup\limits_{H \in \ell_1} (H \cap \gamma_{abc}(H)) \subset \mathbb{P}_n$ is a quadric, find the rank and describe the singular points of this quadric in the both cases: $\dim(L_1 \cap L_2)$ equals either $(n-3)$ or $(n-4)$.

5◇2. Let $S \subset \mathbb{P}_3$ be a surface ruled by all lines tangent to the twisted cubic $C_3 \subset \mathbb{P}_3$. Write down an explicit equation for S, find its degree and all singular points.

5◇3. Let $S \subset \mathbb{P}_5 = \mathbb{P}(S^2V^*)$ be the space of singular conics on $\mathbb{P}_2 = \mathbb{P}(V)$. Show that singular points of S correspond to double lines in $\mathbb{P}(V)$ and $\mathit{Sing}(S)$ coincides with an image of the Veronese embedding $\mathbb{P}(V^*) \overset{v_2}{\hookrightarrow} \mathbb{P}_5$. For non singular $q \in S$, which corresponds to splitted conic $\ell_1 \cup \ell_2 \subset \mathbb{P}(V)$, prove that the tangent space T_qS, for S at q, consits of all conics passing through $\ell_1 \cap \ell_2$.

5◇4* (generalisation of prb. 5◇3). Let $S \subset \mathbb{P}(S^2V^*)$ be the space of singular quadrics on $\mathbb{P}_n = \mathbb{P}(V)$. We write $q \in S$ for a point of S and $Q_q \in \mathbb{P}(V)$ for the corresponding singular quadric.

- **a)** Show that q is non singular point of S iff Q_q is a simple cone (i. e. has just one vertex point).
- **b)** Show that the tangent space $T_qS \subset \mathbb{P}(S^2V^*)$ at non singular $q \in S$ consists of all quadrics passing through the vertex of Q_q.

5◇5 (plane cubics).

- **a)** How many singular points may have a plane cubic curve and what could be their multiplicities?
- **b)** Classify all reducible cubics up to a projective linear isomorphism.
- **c)** Show that irreducible singular cubics are rational and (up to a projective linear isomorphism) are exhausted by $y^2 = x^3$ (nodal cubic) and $y^2 = x^2(x+1)$ (cuspidal cubic).

HINT. rationality may be proved via projection from a singular point; to get nice coordinate representations put the singular point at the origin and draw the axes in such a way that either $y = 0$ would be the cuspidal tangent line or $y = \pm x$ would be the pair of nodal tangent lines.

- **d)** How many tangent lines come to a smooth cubic curve from a generic point on $\mathbb{P}_2$?
- **e)** How many inflection points are there on a smooth cubic?
- **f*)** Show that any non singular cubic may be presented in appropriate affine coordinates by equation $y^2 = x^3 + px + q$.

HINT. see: *C. H. Clemens. A scrapbook of complex curve theory.* Plenum Press. But try to simplify (or to modify) the arguments by your own geometric and/or multilinear arguments

- **g*)** Show that 3 non-inflection tangents which are drown from an inflection point on a smooth cubic meet this cubic in 3 collinear points.

HINT. look at Clemens'es book (*loc. cit.*) and make his arguments to be solid by adding your own details

5◇6. For the plane curves **a)** $(x_0 + x_1 + x_2)^3 = 27\, x_0x_1x_2$ **b)** $(x^2 - y + 1)^2 = y^2(x^2+1)$ find all[2] singular points, compute their multiplicities, look how many branches come to each singularity and what are their geometric tangents.

HINT. to analyze local geometry, blow up the singularity: take affine coordinates (x, y) whose origin is the singularity and substitute $x = \alpha\, t$, $y = \beta\, t$ in the curve equation $f(x, y) = 0$; now $(\alpha\beta)$ parameterizes the line pencil through the origin and $f(\alpha, \beta;\, t) = 0$, as an equation in t, gives the intersection of line $(\alpha\, \beta)$ with the curve; tangent lines are those where the multiplicity of zero root jumps. (Answer for (b): 3 ordinary cusps at … find where!)

[1] problems marked with "star" $*$ are rather honorary than obligatory
[2] including possible singularities at the infinity in (b)