Polynomial ideals[1]

(Nov. 19 — Dec. 3)

6◇1. Give an example of proper non-principal ideal in **a)** $\mathbb{C}[x,y]$ **b)** $\mathbb{Z}[x]$.

6◇2. Let a polynomial f vanish along a hypersurface given in $\mathbb{C}^n$ by a polynomial equation $g=0$. Prove that each irreducible factor of g divides f.

6◇3. Prove that any algebraic set in $\mathbb{C}^2$ is a finite union of points and curves (recall that a curve is a zero set of one polynomial).

6◇4. Let $J=(xy,yz,zx)\subset\mathbb{C}[x,y,z]$. Describe $V(J)\subset\mathbb{A}^3$ and $I(V(J))\subset\mathbb{C}[x,y,z]$. Is it possible to define the same variety by 2 polynomial equations?

6◇5. Find $f\in I(V(J))\setminus J$ for $J=(x^2+y^2-1,y-1)\subset\mathbb{C}[x,y]$.

6◇6. Describe $V(J)\subset\mathbb{A}^3$ and $I(V(J))\subset\mathbb{C}[x,y,z]$ for:
a) $J=(xy,(x-y)z)$ **b)** $J=(xy+yz+zx,x^2+y^2+z^2)$

6◇7. Which of the following three facts about ideals in $k[x_1,x_2,\dots,x_n]$ (k is an arbitrary field) are true? (Prove the true ones and give counter-examples for the other.)
a) $\sqrt{IJ}=\sqrt{I\cap J}$
b) $\sqrt{IJ}=\sqrt{I}\sqrt{J}$
c) $(I=\sqrt{I}\ \&\ J=\sqrt{J})\quad\Rightarrow\quad IJ=\sqrt{IJ}$

6◇8. Let $B\supset A$ be an extension of commutative rings such that B is finitely generated as A-module. Prove that $\mathfrak{m}B\neq B$ for any maximal ideal $\mathfrak{m}\subset A$.

6◇9. Which of the following three rings are Noetherian?
a) $\left\{ f(z)=\dfrac{p(z)}{q(z)}\in\mathbb{C}(z)\ \middle|\ q(z)\neq 0 \text{ for } |z|\leqslant 1\right\}$;
b) power series $f(z)\in\mathbb{C}[[z]]$ converging everywhere on $\mathbb{C}$;
c) $\left\{ f(x,y)\in\mathbb{C}[x,y]\ \middle|\ \dfrac{\partial^{i+j}f}{\partial x^i\partial y^j}=0\quad\forall\, 0\leqslant i+j\leqslant n\right\}$, where $n\in\mathbb{N}$ is fixed.

6◇10*. Show that any finitely generated[2] field is finite (as a set).

6◇11*. Show that the ideal generated in $\mathbb{C}[x_0,x_1,x_2,x_3]$ by all homogeneous polynomials that vanish along the twisted cubic in $\mathbb{CP}_3$
a) is generated by 3 quadratic polynomials
b) can't be generated by 2 polynomials

[1] problems marked with "star" $*$ are rather honorary than obligatory

[2] in absolute sense, i. e. as $\mathbb{Z}$-algebra w. r. t. the action $m\cdot a\overset{\text{def}}{=}\underbrace{a+a+\ \cdots\ +a}_{m\text{ times}}$