Plane curves[1]

(Dec. 3 — Jan. 15)

7◇1. Describe all complex plane projective quintics that have singularities of multiplicity 4 at two given distinct points $a, b \in \mathbb{P}_2$.

HINT. such a quintic have to contain (multiple) line (a, b). Answer: 3-dimensional projective space of equations of the form ... find it!

7◇2. Let $C \subset \mathbb{P}_2$ be a curve with affine equation $x^2y + x\,y^2 = x^4 + y^4$.

a) What kind of singularity has C at the origin?

b) Has C any other singular points (for example, at infinity)?

c) Find the local intersection number at the origin of C with a curve having a simple cusp with tangent line $x = y$.

7◇3. Using Plücker relations, compute the class and the number of inflection points for a smooth plane cubic curve.

7◇4. Let $C \subset \mathbb{P}_2$ be degree d curve, $q \notin C$ lie neither on an inflection tangency nor on a geometric tangency through a singular point of C, and $C_q^{(d-1)}$ be the $(d-1)$-th degree polar of q w. r. t. C. Show that $(C, C_q^{(d-1)})_p$ equals: **a)** 1, if p is smooth; **b)** 3, if p is an ordinary cusp; **c)** $m(m-1)$, if p is an ordinary m-typle node.

HINT. In (a), (b) p is smooth on $C_q^{(d-1)}$ as well and $T_pC_q^{(d-1)} \neq T_pC$ in (a) but $T_pC_q^{(d-1)}$ coincides with the cuspidal tangent in (b). In (c) p is an $(m-1)$-typle point on $C_q^{(d-1)}$ but each geometrical tangency of C at p is transversal to $C_q^{(d-1)}$ and hence intersects it with multiplicity $(m-1)$.

7◇5. Given an irreducible curve $C \subset \mathbb{P}_2$ (maybe singular), show that the closure of a set formed in $\mathbb{P}_2^\times$ by the tangent lines through the smooth points of C is an algebraic curve.

7◇6. Using Plücker relations, list all complex plane quartics with the simplest singularities (ordinary double nodes and cusps) w. r. t. how many cusps, nodes, double tangents and inflexion points may they have. Which of them have to be reducible?

7◇7. Show that smooth plane quartic curve either has a tangent line intersecting the curve just ones with multiplicity 4 or has 28 bitangent lines (touching the curve in two distinct points).

7◇8. Find all lines on the singular projective surface with affine equation $xyz = 1$.

HINT. Show that there are no lines in the initial affine chart

7◇9* (lines on surfaces). Write $\mathbb{P}_N = \mathbb{P}(S^nV^*)$ for the space of degree d surfaces in $\mathbb{P}_3 = \mathbb{P}(V)$ and $Q_P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2V)$ for the Plücker quadric, identified with the set of all lines in $\mathbb{P}_3$. Let $\Gamma \overset{\text{def}}{=} \{\, (S, \ell) \in \mathbb{P}_N \times Q_P \mid \ell \subset S \,\} \subset \mathbb{P}_N \times Q_P \subset \mathbb{P}_N \times \mathbb{P}_5$ be the incidence graph. Prove that:

a) Γ is a projective algebraic variety (i. e. can be defined by a system of bihomogeneous polynomial equations $f_\nu(a, p) = 0$ in $a \in \mathbb{P}_N$ and $p \in \mathbb{P}_5$);

b) fibres of the projection $\Gamma \xrightarrow{\pi_2} Q_P$ are projective spaces of the same dimension; find $\dim \Gamma$.

7◇10*. Using the basic results on the dimensions of projective varieties and their behavior under the projective maps (see lecture 13), show that a general surface $S_d \subset \mathbb{P}_3$ of degree $d \geqslant 4$ does not contain lines but every cubic surface over $\mathbb{C}$ contains lines, typically, a finite set of them.

7◇11*. Let $p \in S \subset \mathbb{P}_3$ be a point on a smooth cubic surface with no lines $\ell \subset S$ passing through p and write Q for the quadratic polar of S w. r. t. p. Consider an apparent contour of S from p, that is a plane projection of $C = Q \cap S$ from p onto a generic plane. Show that this contour is a smooth quartic whose 28 double tangents are projections of 27 lines laying on S and one more line (apropos, what does it come from?).

[1] The ground field is always $\mathbb{C}$ during this task; problems marked with " $*$ " are rather honorary than obligatory