Algebraic manifolds backgrounds[1]

(Dec. 10 — Jan. 15)

8◇1 (Zariski topology). Let $X = Spec_m A$ be affine algebraic set. Check that the sets

$$V(I) = \{\, x \in X \mid f(x) = 0 \quad \forall\, f \in I \,\}$$

produced by all ideals $I \subset A$ satisfy the closed sets axioms of the topology.

8◇2. Prove that any open covering of affine algebraic variety contains a finite sub-covering.

8◇3. Give an example if affine algebraic set X and open $U \subset X$ such that $\mathscr{O}_X(U)$ is not finitely generated as k-algebra.

8◇4. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic sets.
 a) Show that $X \times Y$ is affine algebraic subset in $\mathbb{A}^{n+m}$.
 b) Give $X \times Y \subset \mathbb{A}^{n+m}$ by explicit equations (assuming that the equations for X, Y are known).
 c) Show that $X \times Y$ is irreducible as soon both X, Y are.

8◇5. Prove that the maximal spectrum of a finite dimensional[2] k-algebra is a finite set and deduce from this that any finite morphism has only finite (or empty) fibers.

8◇6. Give an example of regular morphism of affine algebraic sets $X \xrightarrow{\varphi} Y$ such that all fibers of φ are finite (or empty) but φ is not a finite morphism.

8◇7. Prove that a projection of affine hypersurface $V(f) \subset \mathbb{A}^n$ from any point $p \notin V(f)$ onto any hyperplane $H \not\ni p$ is dominant.

8◇8* (Noether's normalization lemma). Show that any affine hypersurface $V(f) \subset \mathbb{A}^n$ admits a finite surjection onto some hyperplane $\mathbb{A}^{n-1} \subset \mathbb{A}^n$.

8◇9. Prove that $\dim(X \times Y) = \dim X + \dim Y$

8◇10. Let $X \xrightarrow{\varphi} Y$ be a regular morphism of algebraic manifolds. Show that isolated[3] points of fibers $\varphi^{-1}(y)$ draw an open subset of X when y runs through Y.
HINT. Use Chevalley's theorem on semi-continuity (lecture 13).

8◇11. Show that an image of a regular dominant morphism contains an open dense subset.

8◇12* (constructivity theorem of Chevalley). Prove that an image of any regular morphism of algebraic varieties is *constructive*, i. e. can be constructed from a finite number of open and closed subsets by a finite number of unions, intersections, and taking complements.

8◇13 (quadratic transformation). Show that the prescription $(t_0 : t_1 : t_2) \longmapsto (t_0^{-1} : t_1^{-1} : t_2^{-1})$ is extended to a rational map $\mathbb{P}_2 \overset{q}{\dashrightarrow} \mathbb{P}_2$ defined everywhere except for 3 points; find these points; clarify how does q act on a triangle (triple of lines) with the vertexes at these 3 points; find $\operatorname{im} q$.

8◇14 (graph of rational map). Let $X \overset{\psi}{\dashrightarrow} Y$ be a rational map defined on open dense $U \subset X$. By the definition, its *graph* $\Gamma_\psi \subset X \times Y$ is the Zariski closure of $\{(x, \psi(x)) \in X \times Y \mid x \in U\}$.
 a) Show that a graph of the natural rational map $\mathbb{A}^{n+1} \dashrightarrow \mathbb{P}_n$, which sends $P \in \mathbb{A}^{n+1}$ to $(OP) \in \mathbb{P}_n$, is isomorphic to the blow up of the origin.
 b*) Try to describe a graph of the quadratic transformation from prb. 8◇13, in particular, describe the fibers of its projections onto the both, source and target, $\mathbb{P}_2$'s.

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[1] the ground field k is always algebraically closed; problems marked with "*" are rather honorary than obligatory
[2] as a vector space over k
[3] a point $p \in M$ is called *isolated* point of a subset $M \subset X$ in a topological space X, if it has an open neighborhood $U \ni p$ such that $U \cap M = \{p\}$