Written Class Test № 1

(December 13, 2002)

Problem 1. Find a condition on 5 lines in $\mathbb{P}_2$ necessary and sufficient for the existence of a unique non-singular conic touching all these lines.

Problem 2. Consider the complex plane quartic[1]

$$(x_0^2 + x_1^2)^2 + 3\,x_0^2 x_1 x_2 + x_1^3 x_2 = 0 \tag{1}$$

a) Find all its singular points over $\mathbb{C}$.

b) Describe their local structure over $\mathbb{C}$ (i. e. geometrical tangents and their intersection multiplicities).

c) Find a rational parameterization for this quartic.

Hint. Use a projection from the singular point onto a line.

d) Show that any irreducible plain quartic with a singularity of multiplicity 3 is rational.

Problem 3.

a) Show that all plane quintics[2] that have an ordinary cusp (of multiplicity 2) at $(0:0:1)$ with the cuspidal tangency $t_1 = 0$ form a projective subspace in the space of all plane quintics.

b) Find the dimension of this subspace.

c) Compute the local intersection multiplicity at $(0:0:1)$ between such a quintic and the quartic (1) from prb. Problem 2.

Problem 4. ([3]) Is there a 2×4 - matrix whose 2×2 - minors are:

a) $\{2, 3, 4, 5, 6, 7\}$

b) $\{3, 4, 5, 6, 7, 8\}$

(If no, explain why, if yes, give an explicit example.)

Hint. Use some congruence reasons instead of fingering all possible permutations...

Problem 5. Prove the Arongold principle: the space of homogeneous degree d polynomials (in several variables) over a field of zero characteristic is linerly spanned by d-th powers of linear forms[4].

Problem 6. Write $S \subset \mathbb{P}(S^2V^*)$ for the hypersurface of singular quadrics in $\mathbb{P}(V)$. Show that a simple quadratic cone[5] with vertex at $p \in \mathbb{P}(V)$ is a non-singular point of S and the tangent hyperplane to S at this point consists of all quadrics passing through p.

Problem 7. Let A be a finitely generated k-algebra. Show that if A is finite dimensional as a vector space over k, then $Spec_m\, A$ is a finite set.

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[1] i. e. a plain curve of degree 4

[2] i. e. plane curves of degree 5

[3] this problem was the 7-th problem of the 3-th home task; if you already have shown me your solutions, then you do not need to do this again, certainly

[4] this implies that any linear assertion about polynomials, e. g. the Taylor formula, is true as soon it holds for d-th powers of all linear forms

[5] i. e. a quadric which has precisely one singular point (called a vertex)