Written Class Test № 2

(February 4, 2003)

Problem 1. Prove that any hypersurface in $\mathbb{A}_n$ admits a finite surjective morphism onto $\mathbb{A}_{n-1}$.
HINT. Use appropriate projection.

Problem 2. Write $\mathbb{P}_N = \mathbb{P}(S^4V^*)$ for the space of quartic hypersurfaces in $\mathbb{P}_3 = \mathbb{P}(V)$ ($\dim V = 4$). Show that all quartics containing a line form a hypersurface in $\mathbb{P}_N$ (in other words, there is a polynomial Φ in the coefficients of variable quartic form $F(x_0 : x_1 : x_2 : x_3)$ such that $\Phi(F) = 0$ iff the quartic $F = 0$ contains a line).

Problem 3. How many triple intersection points[1] have 27 lines on a smooth cubic surface?
HINT. In the local museum of math. things there is a plaster model of a smooth cubic on which all 27 lines are precisely drown. More seriously, this is the most difficult problem in this test; so, don't worry if you will fare from a complete analysis — any progress will be marked highly. What about the Fermat cubic, say?

Problem 4. Show that any nowhere vanishing regular section of the trivial rank r vector bundle over $\mathbb{A}_1$ can included in some system of r regular sections that form a base in each fiber[2].

Problem 5. Consider the standard covering of the Grassmannian $\mathrm{Gr}(m,n)$, of m-dimensional subspaces in k^n, by affine charts U_I consisting of $W \subset k^n$ which are isomorphically projected onto m-dimensional subspace spanned by i_1-th, i_2-th, …, i_m-th basic vectors of k^n along all the other $(n-m)$ basic vectors[3]. Let us present a point $W \in U_I$ by $n\times m$ - matrix $M_I(W)$, whose columns are the coordinates of vectors forming a unique base of W such that $k\times k$ - submatrix situated in i_1-th, i_2-th, …, i_m-th rows is the identity matrix. We consider other $(n-m)\cdot m$ matrix elements (staying outside I-rows) as affine coordinates of W in the chart U_I. Let $S \twoheadrightarrow \mathrm{Gr}(m,n)$ be the tautological vector subbundle of $k^n \times \mathrm{Gr}(m,n)$ whose fiber over a point $W \in \mathrm{Gr}(m,n)$ is the subspace $W \subset k^n$.

a) Construct some trivializing basic sections $s_1^{(I)}, s_2^{(I)}, \dots, s_m^{(I)}$ for S over each U_I and describe corresponding transition matrices $\Phi_{IJ} = \Phi_{IJ}(W)$, which satisfy

$$(s_1^{(I)}, s_2^{(I)}, \dots, s_m^{(I)}) \cdot \Phi_{IJ} = (s_1^{(J)}, s_2^{(J)}, \dots, s_m^{(J)})$$

everywhere in $U_I \cap U_J \subset U_I$.

HINT. Write M_{IJ} for the $m\times m$ - submatrix of M_I situated in J-rows; then M_J is easily expressed through M_I and M_{IJ}.

b) Do the same for the line bundle $D = \Lambda^m S$ and for each its tensor power $D^{\otimes d}$.

c) Prove that any line bundle L over $\mathrm{Gr}(m,n)$ is isomorphic to some $D^{\otimes d}$.

HINT. Write D_{IJ} for determinant $\det M_{IJ}$ of the matrix introduced in the previous hint. The transition function φ_{IJ}, of L, is a rational function in matrix elements of M_I regular and non-vanishing everywhere on U_I except for the zero set of D_{IJ}.

[1] that is, the points where some 3 out of 27 lines are intersecting simultaneously
[2] more honorary (and not obligatory, certainly!) problem is to do the same over $\mathbb{A}_n$
[3] as usually, $I = (i_1, i_2, \dots, i_m)$ runs through all increasing collections of m elements of $\{1, 2, \dots, n\}$