Written examination on algebraic geometry start up course

Notes on marks. Solve five problems below. Complete solution of each problem gives you 12 points. Total sum $\geqslant 45$ points is sufficient for getting the maximal examination mark «A». In fact, each problem is subdivided into several questions; approximate points for answering these questions are shown in brackets near a letter labelling the question.

Problem 1. Describe briefly the Veronese curve $C_n \subset \mathbb{P}_n$, which is the image of Veronese embedding $\mathbb{P}_1 = \mathbb{P}(U) \xhookrightarrow{\nu} \mathbb{P}_n = \mathbb{P}(S^n U)$ and prove that:

- **а) (3 points)** C_n is a smooth algebraic variety
- **б) (3 points)** any $(n+1)$ distinct points of C_n do not lie on a hyperplane
- **в) (6 points)** for any $\psi \in U$ a tangent line $T_{\psi^n} C_n \subset \mathbb{P}_n$ consists of all $f \in S^n U$ divisible by ψ^{n-1} (considerd up to scalar factor)

Problem 2. Is there some 2×4 - matrix with the following set of 2×2 - minors (write down an explicit example of such a matrix or prove that it does not exist):

- **а) (6 points)** $\{2, 3, 4, 5, 6, 7\}$
- **б) (6 points)** $\{3, 4, 5, 6, 7, 8\}$

Problem 3. Let $\mathbb{P}_N = \mathbb{P}(S^2 V^*)$ be the space of quadrics on $\mathbb{P}_n = \mathbb{P}(V)$ and $X \subset \mathbb{P}_N$ be the set of all singular quadrics. Show that

- **а) (3 points)** X is an algebraic variety
- **б) (3 points)** $q \in X$ is smooth iff the corresponding singular quadric $Q \subset \mathbb{P}_n$ has just one singular point
- **в) (6 points)** for any smooth $q \in X$ the tangent space $T_q X \subset \mathbb{P}_N$ consists of all quadrics passing through the singularity of $Q \subset \mathbb{P}_n$

Problem 4. Let k be an arbitrary field and $K = k[x_1, x_2, \dots, x_n]$. Write $\sqrt{I}$ for the radical of an ideal I. Which of the following three propositions are true (prove the true ones and give counter-examples for the other)

- **а) (4 points)** $\sqrt{IJ} = \sqrt{I \cap J}$
- **б) (4 points)** $\sqrt{IJ} = \sqrt{I}\sqrt{J}$
- **в) (4 points)** if I, J are radical ideals, then the product IJ is a radical ideal

Problem 5. Let $C \subset \mathbb{P}_2$ be a curve with affine equation $x^2 y + x\,y^2 = x^4 + y^4$ (in the standard chart).

- **а) (4 points)** What kind of singularity does C have at the origin?
- **б) (4 points)** Find all other singular points of C in $\mathbb{P}_2$ (in particular, at the infinity), if any exist.
- **в) (4 points)** Find the local intersection index at the origin $(C, D)_{(0,0)}$ between C and a curve D that has at the origin a simple cusp with the cuspidal tangency $x = y$.