THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: June 1999

ALGEBRAIC GEOMETRY

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

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1. Segre and Plücker.

a) How many lines meet 4 given mutually skew lines in $\mathbb{P}_3$ over $\mathbb{C}$? Give all possible answers and indicate the most general of them (properties of quadrics may be assumed without proof). **[7]**

b) Let $A = (a_{ij})$ be an $m \times n$ matrix with entries in any field k. Give a definition of the rank of A. If $\operatorname{rk} A = 1$, prove that there exist $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in k^m$ and $(\mu_1, \mu_2, \ldots, \mu_n) \in k^n$ such that $a_{ij} = \lambda_i \mu_j$. **[8]**

c) Explain briefly and without proof the Plücker embedding $\Gamma(2,4) \hookrightarrow \mathbb{P}_5$. Does there exist a complex 2×4 matrix whose 2×2 minors are $\{1, 2, 3, 4, 5, 6\}$ (in some order)? (Give an explicit example or a proof that none exists). **[10]**

Hint: try a congruence argument instead of running over 720 permutations.

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2. Veronese curves. Consider a 2-dimensional vector space U with basis $\{t_0, t_1\}$.

a) Describe briefly the Veronese embedding $\mathbb{P}_1 = \mathbb{P}(U) \hookrightarrow \mathbb{P}_n = \mathbb{P}(S^n U)$ and the Veronese curve $C_n \subset \mathbb{P}_n$. Prove that any $(n+1)$ points of C_n do not lie on a hyperplane. **[7]**

b) Let $\alpha = (\alpha_1 \colon \alpha_0) \in \mathbb{P}_1$ and $p = (\alpha_1 t_0 - \alpha_0 t_1)^n \in C_n$. Show that the tangent line $T_p C_n$ consists of all polynomials $h(t)$ having α as a root of multiplicity $\geqslant (n-1)$. **[8]**

Hint: Differentiate the Veronese embedding ...

c) Show that a plane projection of the twisted cubic C_3 from a point $p \in \mathbb{P}_3$ is either a singular cubic or a non singular conic. Using the standard coordinate presentation of C_3, write down explicit examples of projections whose images are nodal cubic, cuspidal cubic, and smooth conic. How does the projection type depend on the choice of p? **[10]**

Hint: Consider the cases where p lies on C_3, on a tangent line to C_3, and the general case; use a convenient choice of coordinates.

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3. Singular conics. Write $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$ for the space of conics in $\mathbb{P}_2 = \mathbb{P}(V)$ and $\mathcal{S} \subset \mathbb{P}_5$ for the hypersurface of singular conics.

a) Show that all conics which contain 4 given linearly general points form a pencil. Indicate all singular conics in this pencil. How to write down the 1-parameter family of equations that gives all conics of this pencil? **[8]**

b) Show that a locus of singular points on $\mathcal{S}$ coincides with the 2-dimensional Veronese subvariety $\mathcal{V} \subset \mathcal{S}$, which parametrizes the double lines on $\mathbb{P}_2$. **[8]**

c) Let $q = \ell_1 \cup \ell_2 \subset \mathbb{P}_2$ be a reducible conic. Show that $T_q\mathcal{S} \subset \mathbb{P}_5$ consists of all conics passing through the point $\ell_1 \cap \ell_2 \in \mathbb{P}_2$. **[9]**

HINT: try to deal with Laplace relations $A \cdot \widehat{A} = \det(A) \cdot E$, where $\widehat{A} = (\widehat{a}_{ij})$ has $\widehat{a}_{ij} = (-1)^{i+j} \det A_{ji}$ (algebraic complement to a_{ji}); or look at the pencils of quadrics spanned by q and other split conics.

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4. Towards intersection multiplicities.

a) Enumerate (up to a linear automorphism of $\mathbb{P}_2$) all singular plane cubic curves without multiple components. Write down their equations in some convenient coordinates. **[8]**

b) Show that a non singular plane quartic has either a 4-fold tangency or 28 double tangent lines. Formulate (without proof) the Plücker relation you need. **[8]**

c) Let $C \subset \mathbb{P}_2$ be a curve with affine equation $x^2y + x\,y^2 = x^4 + y^4$. What kind of singularity does C have at the origin? Does it have any other singular points (for example, at infinity)? Find the local intersection number at the origin of C with a curve having a simple cusp with tangent line $x = y$. **[9]**

HINT: Use the Zeuthen rule to reduce the calculation to the branches of C.

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5. Lines on surfaces. Write $\mathbb{P}_N = \mathbb{P}(S^nV^*)$ for the space of complex surfaces of degree d in $\mathbb{P}_3 = \mathbb{P}(V)$ and $Q_P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2V)$ for the Plücker quadric, identified with the set of all lines in $\mathbb{P}_3$. Let $\Gamma \overset{\text{def}}{=} \{\, (S,\ell) \in \mathbb{P}_N \times Q_P \mid \ell \subset S \,\} \subset \mathbb{P}_N \times Q_P \subset \mathbb{P}_N \times \mathbb{P}_5$ be the incidence graph.

a) Prove that Γ is a projective algebraic variety, that is, is defined by a system of bihomogeneous polynomial equations $f_\nu(a,p) = 0$ in $a \in \mathbb{P}_N$ and $p \in \mathbb{P}_5$. **[7]**

b) Prove that the fibres of the projection $\Gamma \xrightarrow{\pi_2} Q_P$ are projective spaces of the same dimension and find $\dim \Gamma$. **[8]**

c) Find all lines on the singular projective surface with affine equation $xyz = 1$. **[7]**

HINT: Show that there are no lines in the initial affine chart

d) Assuming known the basic results on the dimensions of projective varieties and their behavior under the projective maps, sketch briefly the proof that a general surface $S_d \subset \mathbb{P}_3$ of degree $d \geqslant 4$ does not contain lines, but that every cubic surface over $\mathbb{C}$ contains lines, typically, a finite set of them. **[7]**
