Lecture 9: Knots, Links, and the Alexander-Conway Polynomial

In this lecture, as in the previous one, we also study curves, but not on the plane $\mathbb{R}^2$ – in space $\mathbb{R}^3$. First, we define knots as polygonal embeddings of $\mathbb{S}^1$ in $\mathbb{R}^3$, define the main equivalence relation (ambient isotopy) and formulate the main (classification) problems of the theory. We then study the famous Reidemeister moves, say a few words about the arithmetic of knots and knot tables, and conclude with the construction of the Conway-Alexander polynomial invariants.

§9.1. Main definitions and problems

A *knot* is a closed oriented polygonal curve in $\mathbb{R}^3$ without self-intersections. A *link* ("zatseplenie" in Russian) is a set of pairwise nonintersecting knots. Examples are shown in Figure 9.1.

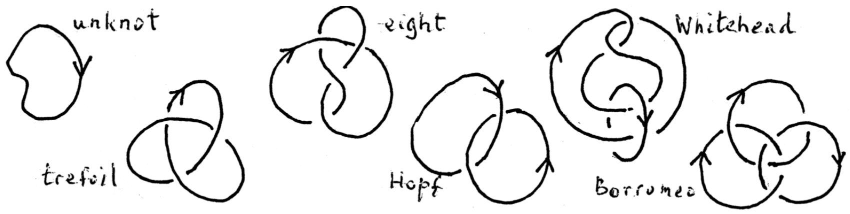


Figure 9.1. Examples of knots and links

Two knots (or two links) are called *equivalent* or *ambient isotopic* if one can be obtained from the other by a finite sequence of *triangular moves* (see Figure 9.2). From now on we consider knots and links up to equivalence, and the words "knot" or "link" mean either concrete curves or the corresponding classes (depending on the context).

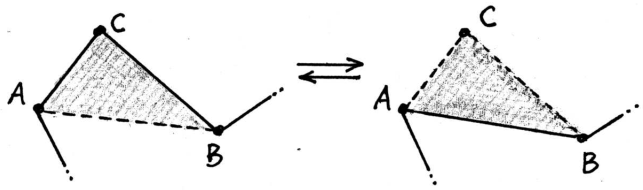


Figure 9.2. Triangular moves

The main problem of knot theory is the *classification problem*: to find an algorithm which determines whether two concrete knots are equivalent or not. An important particular case of this problem is the *unknotting problem*: to find an algorithm which determines whether a concrete knot can be unknotted, i.e., shown to be equivalent to the *trivial knot* or *unknot*, i.e., the circle. Both problems have been solved in principle (Matveev 2005, Hacken 1960), but the solutions are unsatisfactory in the sense that the algorithms are too complicated to be implemented on a computer. Another important problem is the *algebraic classification problem*: to find an algebraic invariant of knots such that two knots are equivalent if and only if they have equal invariants. It has been conjectured that the so-called *Vassiliev invariants* provide a solution to this problem, but there is no proof.

§9.2. Reidemeister moves

We present knots by means of pictures, called *knot diagrams*, which are plane projections of the knot whose only singularities are transversal double points at which the *underpass-overpass* information is given by interrupting one of the branches of the curve (look at Fig.9.1 again). The three types of local modifications of knot diagrams shown in Figure 9.3 are called *Reidemeister moves* and denoted by $\Omega_1, \Omega_2, \Omega_3$.

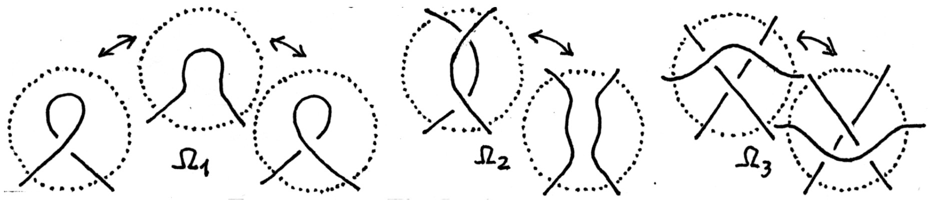


FIGURE 9.3. The Reidemeister moves

Lemma 9.1 [Reidemeister, 1926]. *Two knot diagrams determine the same knot if and only if one can be taken to the other by a finite sequence of Reidemeister moves.*

The proof is a general position argument (which appears, in particular, in the Prasolov–Sossinsky book *Knots, Links, Braids, and 3-Manifolds* on pp.20-22).

§8.3. The arithmetic of knots

If two knots K and K' are given, their *connected sum*, denoted $K\#K'$, is defined as follows: take two knot diagrams representing K and K' and located in two opposite half-planes, and join them as shown in Figure 9.4. The result is well-defined, i.e., does not depend on the choice of the knot diagrams representing K and K' (we will not prove this). A nontrivial knot K is called *prime* if it cannot be decomposed into a nontrivial connected sum, i.e., $K = K'\#K''$ implies that either K' or K'' is the unknot.

FIGURE 9.4. The connected sum of knots

We state the following beautiful *prime knot decomposition theorem* without proof.

Theorem 8.1. [Schubert, 1948] *Any nontrivial knot has a finite decomposition into prime knots, and this decomposition is unique up to order.*

§9.4. Knot tables

It is easy to show that for any fixed k there is a finite number of knots possessing knot diagrams with k crossings.

A knot is called *alternating* if overpasses alternate with underpasses as we move along the curve of the knot. A knot is called *amphichieral* if its mirror image is not equivalent to the knot itself (the first knot in Fig.9.1 is amphichieral, the second one is not.)

The classification problem has been effectively solved for all knots with k crossings, where $k \leq 13$. The answer is usually presented in the form of *knot tables*, which show all prime knots with k crossings or less. (Mirror images of amphichieral knots are not shown.) Such a table for $k = 7$ appears on the next page.

§9.4. The Alexander–Conway polynomial

The *Alexander–Conway polynomial* is a map Δ of the set of link diagrams to the ring of polynomials in one variable x with integer coefficients taking each link diagram L to a polynomial $\Delta_L(x)$ satisfying the following three axioms.

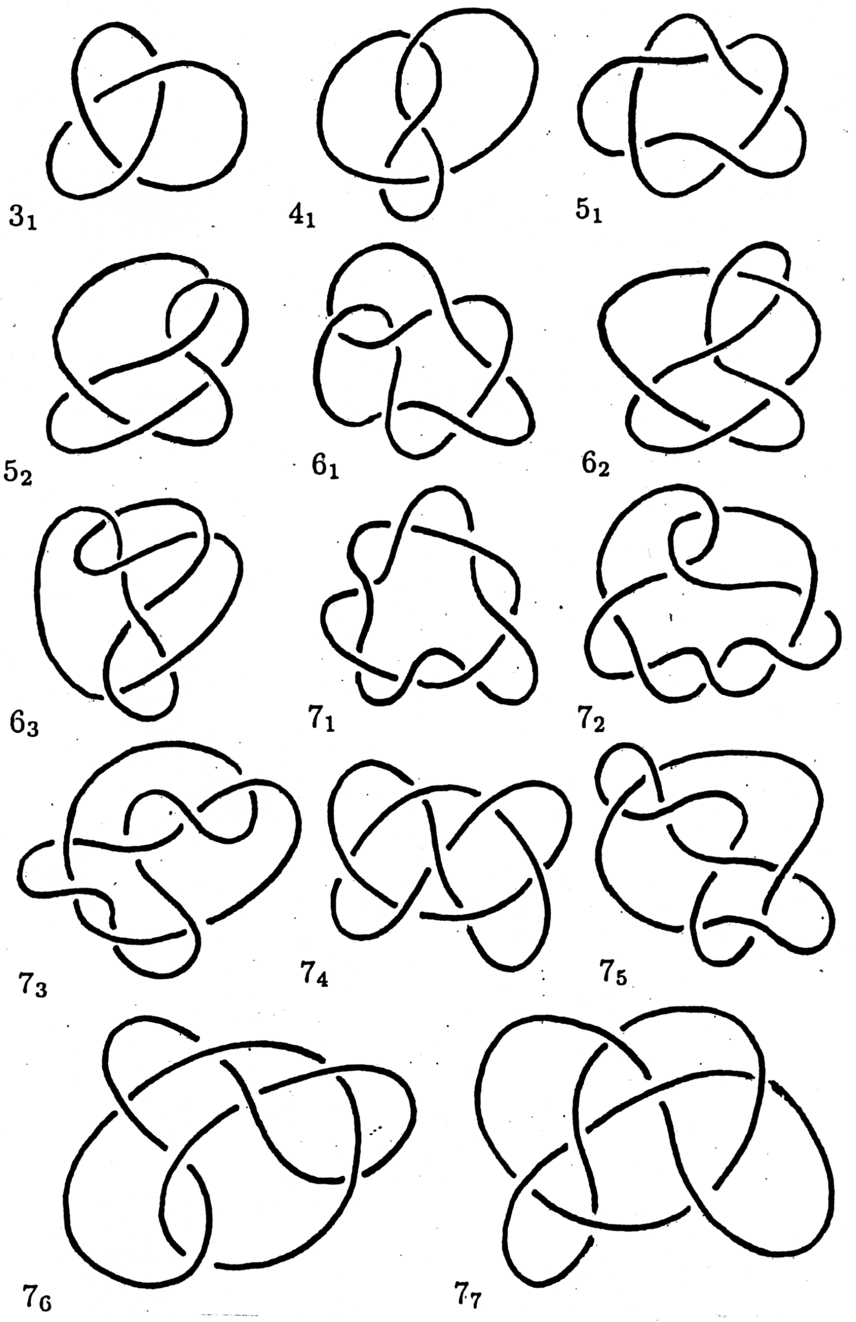


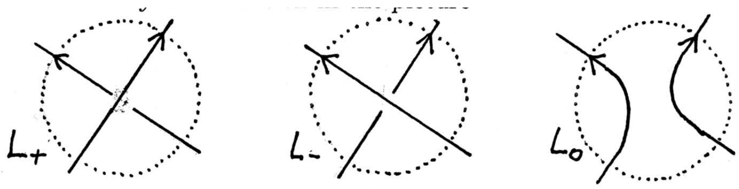
FIGURE 9.5. Prime knots with 7 crossings or less

I. *The map Δ is an invariant, i.e., it yields the same polynomial for equivalent knot diagrams.*

II. *The Alexander–Conway polynomial of the unknot $\bigcirc$ is equal to* 1:

$$\boxed{\Delta_{\bigcirc}(x) = 1}.$$

III. *If L_+, L_-, $L_\circ$ are three link diagrams, identical outside a small disk, inside which they are as shown in the picture*



then $$\boxed{\Delta_{L_+}(x) - \Delta_{L_-}(x) = x \cdot \Delta_{L_\circ}(x)}.$$

These axioms actually provide an algorithm for the computation of the Alexander–Conway polynomial of any knot or link. As an example, we compute the polynomials of the two simplest links.

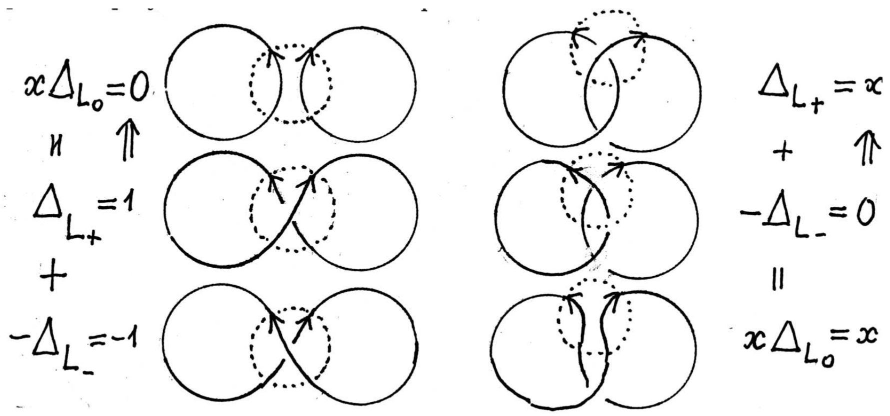


FIGURE 9.7. Computation of $\Delta_L(z)$ for two 2-component links

This computation proves that the two links in the picture are not equivalent, provided, of course, that we have established that a map Δ satisfying axioms I – III actually exists. The latter fact can be proved by using the Reidemeister Lemma, but the proof is not trivial.

§9.5. Problems

9.1. Show that the trefoil and the figure eight knot are not equivalent by computing their Alexander–Conway polynomials.

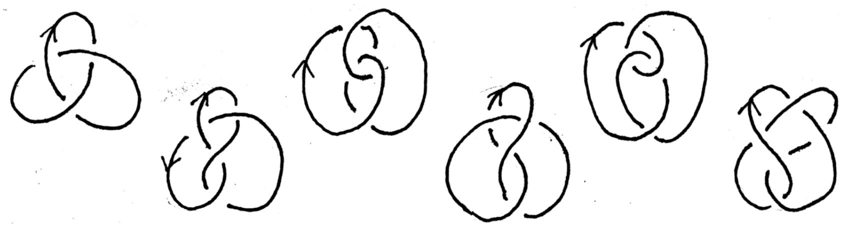
9.2. Compute the Alexander–Conway polynomial of the following links

9.3. Compute the Alexander–Conway polynomial of the knot 7_1 (see the knot table on p.4).

9.4. Prove that the Alexander–Conway polynomial of any two-component link possessing a link diagram with the two components in opposite half-planes is zero.

9.5. Show that the following two knots are equivalent by joining them by a sequence of Reidemeister moves:

9.6. Partition the following eight knot diagrams into equivalence classes.



9.7. Prove that the figure eight knot is amphichieral by constructing its mirror image by Reidemeister moves.

9.8. Show that reversal of the orientation of the trefoil does not change its equivalence class.