Exercises to Lecture I

I.1. (a) Using the ε-δ definition of continuity, give a detailed proof of the fact that composition of two continuous functions is continuous.
(b) Prove that the continuous image of a connected topological space is connected.

I.2. Let $F: \mathbb{R}^2 \to \mathbb{R}$. Suppose the functions $f_{1,x_0}(y) := F(x_0, y)$ and $f_{2,y_0}(x) := F(x, y_0)$ are continues for any x_0, $y_0 \in \mathbb{R}$. Is it true that $F(x, y)$ is continuous?

I.3. Two towns A and B are connected by two roads. Two travellers can walk along these roads from A to B so that the distance between them at any moment is less than or equal to 1 km. Can one traveller walk from A to B and the other from B to A (using these roads) so that the distance between them at any moment is greater than 1 km?

Suppose $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The *distance* from the point x to the subset A is equal to $d(x, A) = \inf\limits_{a \in A} \|x - a\|$.

I.4. (a) Prove that the function $f(x) = d(x, A)$ is continuous for any subset $A \subset \mathbb{R}^n$.
(b) Prove that if the set A is closed, then $f(x) = d(x, A)$ is positive for any $x \notin A$.

I.5. Find the set of points x in $\mathbb{R}^2$ such that $d(x, A) = 1; 2; 3$, where the set A is given by the formula: (a) $x^2 + y^2 = 0$; (b) $x^2 + y^2 = 2$; (c) $x^2 + 2y^2 = 2$; (d) the square of area two.

I.6. Suppose the degrees of all vertices of a connected graph G are all even. Then there exists a path that traverses each edge of G exactly once.

I.7. Prove that any connected planar graph (without loops and double edges) has a vertex of degree not greater than 5.

I.8. Prove that one can color the vertices of any planar graph (without loops) using five colors so that the ends of any edge have different colors.

Let K_n be the graph consisting of n vertices pairwise joined by edges. Let $K_{n,m}$ be the graph consisting of $n + m$ vertices divided into two parts (n vertices in one part and m vertices in the other), the edges of $K_{n,m}$ joining each pair of vertices from different parts.

I.9. (a) Let G be a planar graph such that any face of G is bounded by an even number of edges. Prove that one can color the vertices of G using two colors so that the ends of any edge have different colors.
(b) Let γ be a smooth closed curve with transversal self-intersections. Prove that γ divides the plane into domains so that one can color those domains using two colors (two domains with a common edge must be of different colors).

I.10 (Polygonal Jordan Theorem). Let C be a closed non-self-intersecting broken line (with a finite number of segments) on $\mathbb{R}^2$. Prove that $\mathbb{R}^2 \setminus C$ consists of two connected components and the boundary of each component is C.

I.11. Let a, b, c, d be points of a closed non-self-intersecting broken line C ordered as indicated. Suppose that points a and c are joined by a broken line L_1, points b and d are joined by a broken line L_2 and both broken lines belong to the same connected component defined by C. Prove that L_1 and L_2 have a common point.

I.12. Let G be a *tree*, i.e., a connected graph without cycles. Prove that $v(G) = e(G) + 1$, where $v(G)$ is the number of vertices and $e(G)$ is the number of edges.

I.13 (Euler Formula). Let G be a polygonal planar graph consisting of s connected components each of which is not an isolated vertex. Let G have v vertices and e edges. Using Exercises I.10, I.12 and induction prove that for any embedding of G in the plane the number of faces f is equal to $f = 1+s-v+e$.

I.14. (a) Suppose G is a planar graph without isolated vertices, v_i is the number of its vertices of degree i, f_i is the number of faces with i edges. Prove that $\sum\limits_i (4-i)v_i + \sum\limits_j (4-j)f_j = 4(1+s) \geq 8$, where s is the number of connected components of G.
(b) Prove that if all faces are quadrilaterals, then $3v_1 + 2v_2 + v_3 \geq 8$.
(c) Prove that if the boundary of any face is a cycle containing no less than n edges, then $e \leq n(v-2)/(n-2)$.

I.15. Find and deduce the Euler Formula for convex polyhedra from the Euler formula for planar graphs. (The Euler Formula for convex polyhedra is a relation between numbers of vertices, edges and faces.)

I.16. Prove that the graphs $K_{3,3}$ and K_5 are not planar.