Exercises to Lecture II

II.1. Prove that a constant map is continuous.

For any subsets A, $B \subset \mathbb{R}^n$ the *distance* between A and B is equal to

$$d(A,B) = \inf_{a \in A,\, b \in B} \|a - b\|.$$

II.2. Is it true that $d(A,C) \le d(A,B) + d(B,C)$?

II.3. Let $A \subset \mathbb{R}^n$ be a closed subset, let $C \subset \mathbb{R}^n$ be a compact subset. Prove that there exists a point $c_0 \in C$ such that $d(A,C) = d(A,c_0)$. Further, prove that if the set A is also compact, then there exists a point $a_0 \in A$ such that $d(A,C) = d(a_0,c_0)$.

II.4. Prove that any closed subspace of a compact space is compact.

II.5. Prove that the topology of $\mathbb{R}^n$ has a countable base.

II.6. Introduce a “natural” topology on
(a) the set $\mathrm{Mat}(m,n)$ of matrices of size $n \times m$;
(b) the real projective space $\mathbb{R}P(n)$ of dimension n;
(c) the Grassmannian $G(k,n)$, i.e., the set of k-dimensional planes containing the origin of n-dimensional affine space;
(d) the set of solutions $p = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4$ of the following system of 2 equations: $x_1^2 + x_2^3 + x_3^4 + x_4^5 = 1$ and $x_1x_2x_3x_4 = -1$;
(e) the set of all polynomials of degree n with leading coefficient 1.

II.7. Construct a connected but not path connected topological space. [*Hint*: Use some construction with the complement of the graph of the function $\sin(1/x)$, $x > 0$.]

II.8. (a) Is the topological space $\mathrm{GL}(n)$ connected?
(b) Prove that the topological space $\mathrm{SO}(3)$ is connected.
(c) Prove that the topological space $\mathrm{GL}(3)$ consists of two connected components.

II.9. (a) Prove that $d(x,y) = \max_{i=1,\dots,n} |x_i - y_i|$, where $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$, is a metric in $\mathbb{R}^n$.
(b) Prove that $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ is a metric in $\mathbb{R}^n$.
(c) Draw some ε-neighborhood of the point $(0,0,\dots,0)$ in the metrics defined in (a) and (b).

II.10. Prove that any metric space is Hausdorff.

II.11. Give some example of a non-Hausdorff space.

II.12. Let X be a Hausdorff space. Prove that for any two distinct points $x, y \in X$ there exists a neighborhood $U \ni x$ such that its closure does not contain the point y.

II.13. Let C be a compact subspace of a Hausdorff space X. Let $x \in X \setminus C$. Prove that the point x and the set C have disjoint neighborhoods.

II.14. Prove that any two disjoint compact subsets of a Hausdorff space have disjoint (open) neighborhoods.

II.15. A plane hinge mechanism consists of two rods of equal size with one extremity at a fixed hinge and the other extremity free. Find the configuration space of this system
(a) if the rods can occupy the same position and pass one over the other;
(b) if they cannot.

Let $\langle l_1, l_2, \dots, l_{n-1}; d\rangle$ be a plane hinge mechanism that consists of n rods, one of which is fixed and the other rods (together with the fixed rod) form a closed polygonal line, with fixed rod of length d and moving rods of lengths $l_1, l_2, \dots, l_{n-1}$ numbered successively.

II.16. Find the configuration spaces of the following quadrangles: (a) $\langle 1,1,1;2.9\rangle$; (b) $\langle 1,1,1;1\rangle$; (c) $\langle 2,3,2;3\rangle$.

II.17. Find the configuration spaces of (a) $\langle 1,1,1,1;3.9\rangle$; (b) $\langle \underbrace{1,1,\dots,1}_{n}; n-0.1\rangle$.

II.18. Find the configuration spaces of (a) $\langle 6,2,2,6;6\rangle$; (b) $\langle 2,1,1,2;2\rangle$.