Exercises to Lecture III

III.1. Prove that $D^n/\partial D^n \approx S^n$.

III.2. Prove that the space $S^1 \times S^1$ is homeomorphic to the space obtained by the following identification of points of the square $0 \le x, y \le 1$ belonging to its sides: $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$. (This space is called the *torus*.)

III.3. Let $I = [0,1]$. Prove that the space $S^1 \times I$ is not homeomorphic to the Möbius band.

III.4. Prove that the following spaces are homeomorphic:
(a) the set of lines in $\mathbb{R}^{n+1}$ passing through the origin;
(b) the set of hyperplanes in $\mathbb{R}^{n+1}$ passing through the origin;
(c) the sphere S^n with identified diametrically opposite points (every pair of diametrically opposite points is identified);
(d) the disc D^n with identified diametrically opposite points of the boundary sphere $S^{n-1} = \partial D^n$.

III.5. Prove that the following spaces are homeomorphic:
(a) the set of complex lines in $\mathbb{C}^{n+1}$ passing through the origin;
(b) the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with identified points of the form λx for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$ (for any fixed point $x \in S^{2n+1}$);
(c) the disc $D^{2n} \subset \mathbb{C}^n$ with identified points of the boundary sphere $S^{2n-1} = \partial D^{2n}$ of the form λx for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$ (for any fixed point $x \in S^{2n-1}$).

III.6. Prove that $CD^n \approx D^{n+1}$ and $\Sigma D^n \approx D^{n+1}$.

III.7. Prove that $\mathbb{R}P^1 \approx S^1$ and $\mathbb{C}P^1 \approx S^2$.

III.8. Prove that $CS^n \approx D^{n+1}$ and $\Sigma S^n \approx S^{n+1}$.

III.9. Is it true (for arbitrary CW-complex) that (a) $X * Y \approx Y * X$; (b) $(X*Y)*Z \approx X*(Y*Z)$; (c) $C(X*Y) \approx CX*Y$; (d) $\Sigma(X*Y) \approx \Sigma X*Y$?

III.10. Prove that $S^n * S^m \approx S^{n+m+1}$.

III.11. Prove that $\mathbb{R}^n \setminus \mathbb{R}^k \approx S^{n-k-1} \times \mathbb{R}^{k+1}$.

III.12. Prove that $S^{n+m-1} \setminus S^{n-1} \approx \mathbb{R}^n \times S^{m-1}$. (We suppose that the disposition of S^{n-1} in S^{n+m-1} is standard.)

III.13. Let $S^p \vee S^q = (S^p \times \{*\}) \cup (\{*\} \times S^q) \subset S^p \times S^q$. Prove that $(S^p \times S^q)/(S^p \vee S^q) \approx S^{p+q}$.

III.14. Prove that the sphere S^2 is a CW-complex.

III.15. Prove that the torus T^2 is a CW-complex.

III.16. Prove that the sphere S^n is a CW-complex.

III.17. Prove that the real projective space $\mathbb{R}P^n$ is a CW-complex.

III.18. Prove that the complex projective space $\mathbb{C}P^n$ is a CW-complex.

III.19. (a) Find an example of a complex that satisfies the W-axiom, and does not satisfy the C-axiom.
(b) Suppose some complex satisfies the C-axiom, is it true that it satisfies the W-axiom?

III.20. Prove that the configuration spaces of the hinge mechanisms described in Exercises II.15—II.18 are all CW-complexes.