Exercises to Lecture IV

IV.1. (a) Attaching a handle is the same as attaching a horseshoe-shaped cylinder
(b) Attaching a Klein bottle is the same as attaching a "twisted" cylinder

IV.2. The projective plane is
(a) the Möbius strip with a disk attached;
(b) the sphere S^2 with antipodal points identified;
(c) the disk D^2 with diametrically opposed points identified.

IV.3. The Klein bottle is
(a) the double of the Möbius strip;
(b) the sphere with two holes with two Möbius strips attached;
(c) the connected sum of two projective planes.

IV.4. (a) Consider the topological space of straight lines in the plane. Prove that this space is homeomorphic to the Möbius band without boundary.
(b) Consider the topological space of *oriented* straight lines in the plane. Prove that this space is homeomorphic to the cylinder without boundary.

IV.5. Show that a punctured tube from a bicycle tire can be turned inside out. (More precisely, this would be possible if the rubber from which the tube is made were elastic enough.)

IV.6 (Polygonal Schoenflies Theorem). A closed polygonal line in the plane bounds a domain whose closure is the disk D^2.

IV.7 (Polygonal Annulus Theorem). Two closed polygonal lines in the plane, one of which encloses the other, bound a domain whose closure is the annulus $S^1 \times [0,1]$.

IV.8. (a) The two surfaces-with-holes obtained from the same closed triangulated surface by removing two different open 2-simplices from it are homeomorphic.
(b) Show that the connected sum of surfaces is well defined.

IV.9. Prove that $T^2 \# \mathbb{R}P^2 \approx 3\mathbb{R}P^2$.

IV.10. (a) Prove that Kl#Kl is homeomorphic to the Klein bottle with one handle attached.
(b) Prove that $\mathbb{R}P^2 \#$ Kl is homeomorphic to the projective plane with one handle attached.

IV.11. Prove that if a surface M_1 is nonorientable, then for any surface M_2 the surface $M_1 \# M_2$ is nonorientable.

IV.12. How many different surfaces is it possible to glue (by identifying sides) starting with (a) square; (b) hexagon; (c) octagon.

IV.13. (a) Prove that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.
(b) Prove that $\chi(mT^2) = 2 - 2m$ and $\chi(n\mathbb{R}P^2) = 2 - n$.

IV.14. Prove that a closed orientable surface is not homeomorphic to a closed nonorientable surface.

IV.15. Prove that the Euler characteristics of two homeomorphic closed surfaces are equal.

IV.16. Prove that a simplicial neighborhood of a tree is homeomorphic to the disc.

IV.17. Find χ(Kl).

IV.18. Let M_1 and M_2 be two homeomorphic surfaces, prove that $\chi(M_1) = \chi(M_2)$.

IV.19. Determine the type of the surface shown in Fig. IV.1 (a) for $n = 3$; (b) for arbitrary $n \geq 2$.

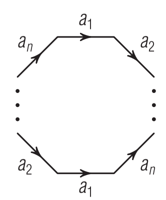

Fig. IV.1: Determine the type of this surface

IV.20. (a)* Prove that any closed nonorientable surface is homeomorphic to one of the surfaces in the following list:

$$\begin{array}{ll} \mathbb{R}P^2 & \text{(projective plane)}, \\ \mathbb{R}P^2 \# \mathbb{R}P^2 & \text{(Klein bottle)}, \\ \dots\dots\dots & \\ \mathbb{R}P^2 \# \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2, & \\ \dots\dots\dots & \end{array}$$

(b) Any two distinct surfaces in the list are not homeomorphic.

IV.21. Consider the quotient space $(S^1 \times S^1)/(x,y) \sim (y,x)$. This space is a surface. Which one?

IV.22. Prove that the boundary of $\mathrm{Mb}^2 \times I$ is the Klein bottle.

IV.23. Prove that on the sphere with g handles, the maximal number of nonintersecting closed curves not dividing this surface is equal to g.