Exercises to Lecture V

V.1. (a) Suppose that $X = X_1 \cup \ldots \cup X_n$, where the sets $X_1, \ldots, X_n$ are closed. Consider a map (not necessarily continuous) $f \colon X \to Y$ and its restrictions $f_i = f|_{X_i}$. Prove that the map f is continuous iff every map f_i is continuous.
(b) Prove the same assertion for open sets $X_1, \ldots, X_n$.

V.2. (a) Prove that if the image of a map $f \colon X \to S^1$ is not the whole space S^1 (i.e., the map is not onto), then f is homotopic to a constant map.
(b) Prove that if a map $f \colon X \to S^n$ is not onto, then f is homotopic to a constant map.

V.3. Prove that the spaces $S^1 \vee I$ and S^1 are homotopy equivalent.

V.4. Prove that the following spaces are homotopy equivalent: (a) the sphere S^2 with two points identified; (b) the union of the sphere S^2 and one of its diameters; (c) $S^1 \vee S^2$.

V.5. Prove that the sphere with g handles from which a point has been removed is homotopic to the wedge product of n copies of the circle and find n.

V.6. Prove that the spaces $\mathbb{R}^3 \setminus S^1$ and $S^2 \vee S^1$ are homotopy equivalent.

V.7. Let X be the space $\mathbb{R}^3$ from which n copies of S^1 are deleted (all circles S^1 are unknotted and pairwise nonlinked). Prove that X is homotopy equivalent to the wedge product of n copies of the space $S^2 \vee S^1$.

V.8. Let L be a two circles in $\mathbb{R}^3$ linked in the simplest way. Prove that the spaces $\mathbb{R}^3 \setminus L$ and $S^2 \vee T^2$ are homotopy equivalent.

V.9. Prove that the following assertions are equivalent:
(a) Any continuous map $f : D^n \to D^n$ has a fixed point.
(b) There is no retraction $r : D^n \to S^{n-1}$.
(c) Let $v(x)$ be a continuous vector field on D^n such that $v(x) = x$ for any point $x \in S^{n-1}$. Then $v(x) = 0$ for some point $x \in D^n$.

V.10. Prove that A is a retract of X if and only if any continuous map $f \colon A \to Y$ can be extended to X.

V.11. Prove that if any continuous map $X \to X$ has a fixed point and A is a retract of X, then any continuous map $A \to A$ also has a fixed point.

V.12. Let S^∞ be the set of points $(x_1, x_2, \ldots) \in \mathbb{R}^\infty$ such that only finite number of coordinates are nonzero and $\sum x_i^2 = 1$. It is a metric space, so it has a natural topology. Prove that the space S^∞ is contractible. [*Hint*: prove that the identity map is homotopic to the map $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$.]