Exercises to Lecture VI

VI.1. On the complex plane consider the vector field $v(z) = \frac{z^n}{|z|^{n-1}}$ for $z \neq 0$, $v(0) = 0$. Find the index of the singular point of this field (for any integer n).

VI.2. Prove that the index of the curve γ is equal to the sum of indices of the singular points that it encircles.

VI.3. Suppose that two vector fields v and w are given on a closed non-self-intersecting curve in such a way that at any point X the vectors $v(X)$ and $w(X)$ do not point in exactly opposite directions. Prove that the indices of γ with respect to these vector fields are equal.

VI.4. Using Exercise VI.3, prove that any polynomial $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ with complex coefficients has at least one complex root.

VI.5. Let us say that a vector field v is *even* if $v(x) = v(-x)$ and *odd* if $v(x) = -v(-x)$. Prove that the index of the point O for an even field is even and is odd for an odd field.

VI.6. A closed self-intersecting curve divides the plane into several regions. By choosing a point O in each region, we can assign to the region the number of revolutions performed by the vector $\overrightarrow{OX}$ when the point X goes around the curve. Prove that if two regions have a common boundary, then the two numbers for the two regions differ by 1.

VI.7. On the boundary circles of an annulus consider a vector field such that the vectors are tangent to the circles and the vectors at any two corresponding points of the circles have opposite directions. Extend this vector field to a vector field without singular points on the entire annulus.

VI.8. Prove that a vector field given on the boundary circles of an annulus can be extended to a vector field without singular points on the entire annulus if and only if the indices of two circles are equal.

VI.9. Let f be a smooth function on the plane. Prove that the index of an isolated singular point of the vector field $v = \operatorname{grad} f$
(a) can be equal to $1, 0, -1, -2, \ldots$ and
(b)* can not be equal to the other integers.

VI.10. On the torus construct a vector field without singular points.

VI.11. On the Klein bottle construct a vector field without singular points.

VI.12. On the sphere construct a vector field with one singular point.

VI.13. On the projective plane construct a vector field with one singular point.

VI.14. On the sphere with two handles construct a vector field with one singular point.

VI.15. To each point X on the sphere $S^2 \subset \mathbb{R}^3$ a nonzero vector $v(X)$ in space is assigned. The vector depends continuously on the point of the sphere, but is not necessarily tangent to it. Prove that at least one of the vectors $v(X)$ is perpendicular to the tangent plane to the sphere at the point X.

VI.16. Let $f: S^2 \to S^2$ be a continuous map. Prove that there exist a point $x \in S^2$ such that $f(x) = \pm x$.