Exercises to Lecture VII

VII.1. Prove that the fundamental group of the wedge product of n circles is isomorphic to the free group with n generators.

VII.2. Prove that the group $\pi_1(nT^2)$ is generated by elements $a_1, b_1, \ldots, a_n, b_n$ obeying to the unique relation $\prod_{i=1}^{n}(a_i b_i a_i^{-1} b_i^{-1}) = 1$.

VII.3. Prove that the group $\pi_1(n\mathbb{R}P^2)$ is generated by elements $a_1, \ldots, a_n$, obeying to the unique relation $a_1^2 \ldots a_n^2 = 1$.

VII.4. (a) Prove that if $G = \pi_1(nT^2)$, then $G/G' \cong \mathbb{Z}^{2n}$. (Here G' is a *commutant*, i.e. G' is a subgroup generated by all elements of the type $aba^{-1}b^{-1}$ for $a, b \in G$.)
(b) Prove that if $G = \pi_1(n\mathbb{R}P^2)$, then $G/G' \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2$.

VII.5. Prove that $\pi_1(S^n) = 0$ for $n \geq 2$.

VII.6. Prove that $\pi_1(\mathbb{C}P^n) = 0$.

VII.7. Prove that the fundamental group of the surface nT^2 with $k \geq 1$ deleted discs is the free group of rank $2n + k - 1$.

VII.8. Prove that the fundamental group of the surface $n\mathbb{R}P^2$ with $k \geq 1$ deleted discs is the free group of rank $n + k - 1$.

VII.9. Suppose that X is the Möbius band, A is its boundary. Prove that A is not a retract of X.

VII.10. Prove that any finite and connected CW-complex is homotopy equivalent to CW-complex with only one vertex e^0.