Lectures on homotopical and model categories

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April 5, 2012

Contents

1 Introduction

Thanks to Paul Taylor for his diagram package.

2 Motivation

Algebraic topology has grown to an enormous subject today, and most mathematicians are familiar with the notion of homotopy. It is often the case that a couple of spaces X, Y are not really meant to be distinguished when it comes to the invariants we want to extract from them. One such invariant is i-th homotopy group $\pi_i(X, x)$ of a topological space X with a basepoint x. This is defined as the set of all maps[1] $(S^i, 1) \to (X, x)$ modulo the equivalence relation of homotopy: two maps $f, g : Z \to X$ between Z and X are homotopic if there is a map $H : Z \times [0,1] \to X$ such that $H(z, 0) = f(z)$ and $H(z, 1) = g(z)$.

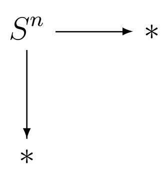
One observes that if two spaces X, Y are homotopic, that is, if there is a couple of maps $f : X \to Y$ and $g : Y \to X$, such that their compositions are homotopic to identity maps, then $\pi_i(X, x) \cong \pi_i(Y, f(x))$ for $i \in \mathbb{N} = \{0, 1, 2, ...\}$ and some choice of a homotopy f. More generally, one defines a map $f : X \to Y$ to be a *weak homotopy equivalence* if

$$\pi_i(f) : \pi_i(X, x) \to \pi_i(Y, f(x))$$

is an isomorphism for all $i \in \mathbb{N}$ and $x \in X$.

Homotopy equivalences Heq (that is, maps admitting homotopy inverses) and weak homotopy equivalences Weq are of great importance in topology. For several reasons one chooses weak homotopy eqivalences to work with, and both classes of maps coincide on 'good' topological spaces (e.g. CW-complexes). Regardless of the class of maps chosen, one is stuck with the issue of *inverting* the elements of this class. This problem naturally arises due to the fact that these maps induce isomorphisms between all homotopy-theoretic invariants of two (weakly) homotopic spaces.

Some usual constructions result in different spaces when the initial data is replaced by a homotopically equivalent one. Consider, for example, the following diagram of spaces:



We might glue[2] the two points along S^n to obtain, unsurprisingly, a point $*$. However, we might

[1] That is, the maps $S^i \to X$ which send the distinguished point $1 \in S^i$ to x

[2] that is, take a colimit of this diagram

consider a diagram

$$\begin{array}{ccc} S^n & \longrightarrow & D^{n+1} \\ \downarrow & & \\ D^{n+1} & & \end{array}$$

where S^n is included as the boundary of D^{n+1}'s, and then glue and obtain S^{n+1} (in fact, another way to view this is that we replaced a point by CS^n, the cone over S^n, in this diagram and obtained the space ΣS^n, the *suspension* of S^n, which is equivalent to S^{n+1}). The construction just outlined is an example of a *homotopy colimit.* This is a homotopically correct replacement of a colimit functor (see below), or, more precisely, the left derived functor of a colimit functor.

In these lectures we shall be concerned with the general machinery applicable to studying the formal notion of homotopy. We shall not restrict ourselves to topological spaces: it will be possible to apply this machinery to a whole lot of situations in mathematics. This machinery is called *Model Category Theory.* It is true that today there are other techniques, e.g. Joyal and Lurie's *quasicategories*, which aim at solving the same problem and are sometimes more 'natural' than model categories. However, it is somewhat ironic that in order to understand quasicategories one needs a great deal of model category theory (this is why [3] is among the references for this course), so model categories still remain a tool one should know about in order to study abstract homotopy theory.

3 Basics of Category Theory

3.1 Categories

We begin our course by discussing the mathematically appropriate framework for homotopical algebra. This inculdes category theory (which is not surprising), with examples aimed at illuminating the most known 'categories with weak equivalences', that is, complexes , categories of (small) categories, topological spaces and, most importantly, simplicial sets (the latter will be discussed in much greater generality after the introduction of Model Categories).

Definition 3.1. A category $\mathcal{C}$ consists of the following data:

1. A set of *objects* $Ob\,\mathcal{C}$. For the sake of brevity, we shall write $X \in \mathcal{C}$ for $X \in Ob\,\mathcal{C}$.

2. For every couple of objects $X, Y \in \mathcal{C}$, there is a set of *morphisms* $\mathcal{C}(X,Y)$. We also might denote this set by more traditional $Hom_{\mathcal{C}}(X,Y)$ or even $Hom(X,Y)$ when the name of the category we consider is clear from the context. An element of $\mathcal{C}(X,Y)$ will be portrayed as

$$X \longrightarrow Y$$

3. For every triple of objects X, Y, Z of $\mathcal{C}$, there is a *composition map*

$$\circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z),$$

$$(f,g) \mapsto g \circ f.$$

4. For every object X of $\mathcal{C}$, there is a distinguished element $1_X \in \mathcal{C}(X,X)$.

These data are subject to the following:

- The composition is associative:

$$(h \circ g) \circ f = h \circ (g \circ f)$$

  whenever the triple h, g, f is composable. In other words, the diagram

$$\begin{array}{ccc} \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \times \mathcal{C}(Z,T) & \xrightarrow{\circ \times id} & \mathcal{C}(X,Z) \times \mathcal{C}(Z,T) \\ {\scriptstyle id \times \circ} \downarrow & & \downarrow {\scriptstyle \circ} \\ \mathcal{C}(X,Y) \times \mathcal{C}(Y,T) & \xrightarrow{\quad \circ \quad} & \mathcal{C}(X,T) \end{array}$$

  commutes.

- The distinguished elements $1_X \in \mathcal{C}(X,X)$ serve, for every X, as units for the composition: for every $f : X \to Y$,

$$f \circ 1_X = 1_Y \circ f = f.$$

The examples of categories are in abundance.

Definition 3.2. Given a category $\mathcal{C}$, its *opposite category* $\mathcal{C}^{op}$ is defined as follows:

- $Ob\, \mathcal{C}^{op} = Ob\, \mathcal{C}$,
- $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$.

The composition and units in $\mathcal{C}^{op}$ are naturally induced from those in $\mathcal{C}$.

Definition 3.3. Given two categories $\mathcal{C}, \mathcal{D}$, the *product* of $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$, such that $Ob\, \mathcal{C} \times \mathcal{D} = Ob\, \mathcal{C} \times Ob\, \mathcal{D}$ and $\mathcal{C} \times \mathcal{D}((X_1,Y_1),(X_2,Y_2)) = \mathcal{C}(X_1,X_2) \times \mathcal{D}(Y_1,Y_2)$. That is, to give a map $\alpha : (X_1,Y_1) \to (X_2,Y_2)$ in $\mathcal{C} \times \mathcal{D}$ is the same as to give a couple of maps $f : X_1 \to X_2$ in $\mathcal{C}$ and $g : Y_1 \to Y_2$ in $\mathcal{D}$ (in this case we write $\alpha = (f,g)$). The composition is induced from that of $\mathcal{C}$ and $\mathcal{D}$.

Example 3.4. A category M with a single object $\bullet$ is essentially specified by the set $M(\bullet,\bullet)$. This set is equipped with an associative binary operation with unit. Such a set is called *monoid*. In particular, for every group G, there is a category $\mathbf{B}G$ with the single object $\bullet$ and $\mathbf{B}G(\bullet,\bullet) = G$.

Example 3.5. The following examples could be called 'concrete' categories:

- The category **Set**: objects are sets, for $X, Y \in \mathbf{Set}$, $\mathbf{Set}(X, Y)$ is the set of all maps between X and Y.

- The category $\mathbf{Vect}_k$: objects are vector spaces over a field k, for $V, W \in \mathbf{Vect}_k$, $\mathbf{Vect}_k(V, W)$ is the set of all *linear* maps between V and W.

- The category **Top**: objects are topological spaces, for $X, Y \in \mathbf{Top}$, $\mathbf{Top}(X, Y)$ is the set of all *continious* maps from X to Y.

- The category $\mathbf{DGVect}_k$: objects are (countably infinite) diagrams of vector spaces over k

$$\dots \xrightarrow{d^{i-2}} V^{i-1} \xrightarrow{d^{i-1}} V^i \xrightarrow{d^i} V^{i+1} \xrightarrow{d^{i+1}} \dots$$

  (which we shall denote for brevity by $V^\bullet$) with the property that for every $i \in \mathbb{Z}$, $d^{i+1} \circ d^i = 0$. For $V^\bullet, W^\bullet \in \mathbf{DGVect}_k$, $\mathbf{DGVect}_k(V^\bullet, W^\bullet)$ is the set of all diagrams

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d_V^{i-2}} & V^{i-1} & \xrightarrow{d_V^{i-1}} & V^i & \xrightarrow{d_V^i} & V^{i+1} & \xrightarrow{d_V^{i+1}} & \dots \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \dots & \xrightarrow{d_W^{i-2}} & W^{i-1} & \xrightarrow{d_W^{i-1}} & W^i & \xrightarrow{d_W^i} & W^{i+1} & \xrightarrow{d_W^{i+1}} & \dots \end{array}$$

  such that $\forall i \in \mathbb{Z}$, $d_W^i \circ f^i = f^{i+1} \circ d_V^i$. The composition of $f = \{f^i\}_{i\in\mathbb{Z}} : U^\bullet \to V^\bullet$ and $g = \{g^i\}_{i\in\mathbb{Z}} : V^\bullet \to W^\bullet$ is done 'pointwise': $g \circ f = \{g^i \circ f^i\}_{i\in\mathbb{Z}} : U^\bullet \to W^\bullet$.

- We can replace vector spaces in the above examples by modules over a ring A to obtain categories $\mathbf{Mod}_A$ and $\mathbf{DGMod}_A$.

3.2 'Size issues' and universes

One might object that the above examples are ill-defined: there is no such thing as 'the set of all sets' (at least in the well-accepted axiomatics of set theory). There are various ways of making this a non-problem. We choose one of them and assume that

(U) every set is an element of some universe,

where a *universe* is, so to speak, a set of 'sufficiently small sets'. The axioms of it are summarized in the definition below:

Definition 3.6. A *universe* $\mathbb{U}$ is a set of sets (called $\mathbb{U}$-*sets*), such that

I. If $x \in \mathbb{U}$ and $y \in x$, then $y \in \mathbb{U}$,

II. $x, y \in \mathbb{U} \Rightarrow \{x, y\} \in \mathbb{U}$,

III. If $I \in \mathbb{U}$ and $\forall\, i \in I, x_i \in \mathbb{U}$, then $\cup_{i \in I} x_i \in \mathbb{U}$,

IV. If $x \in \mathbb{U}$, then the set of all subsets $P(x)$ of x belongs to $\mathbb{U}$,

V. $\mathbb{N} \in \mathbb{U}$, where $\mathbb{N}$ is the set of natural numbers.

One can prove that

Proposition 3.7. *For a universe $\mathbb{U}$, the following is true:*

VI. $x \in \mathbb{U}, y \subseteq x \Rightarrow y \in \mathbb{U}$

VII. *If $I \in \mathbb{U}$ and $\forall\, i \in I, x_i \in \mathbb{U}$, then $\prod_{i \in I} x_i \in \mathbb{U}$ and $\coprod_{i \in I} x_i \in \mathbb{U}$.*

VIII. $x, y \in \mathbb{U} \Rightarrow x^y = \{f : y \to x\} \in \mathbb{U}$.

IX. *Given any set of universes $\{\mathbb{U}_i\}$, the intersection $\cap_i \mathbb{U}_i$ is a universe,*

X. $x \in \mathbb{U} \Rightarrow x \subseteq \mathbb{U}$.

Proof. See e.g. 'Grothendieck Universe' on nLab. □

An opposite of property (X) does not hold: this would, for example, imply $\mathbb{U} \in \mathbb{U}$, and hence, by the property (IV), $P(\mathbb{U}) \subseteq \mathbb{U}$, which is not possible due to the fact that $P(\mathbb{U})$ has cardinality greater than $\mathbb{U}$.

Moreover,

Corollary 3.8. *For any universe $\mathbb{U}$ there exists (due to (IX)) a minimal universe $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$.*

Armed with the metamathematics of universes, we can now deal with the size issues in the following way.

Definition 3.9. Let $\mathbb{U} \in \mathbb{V}$ be a fixed pair of a universe and its successor.

- A $\mathbb{U}$-category $\mathcal{C}$ has $Ob\, \mathcal{C} \subseteq \mathbb{U}$, and for all objects X, Y $\mathcal{C}(X, Y) \in \mathbb{U}$.
- A *small* $\mathbb{U}$-category I, or $\mathbb{U}$-small category, is a $\mathbb{U}$-category such that $Ob\, I \in \mathbb{U}$.
- The word 'category' will mean small $\mathbb{V}$-category.

All this is useful for a 'working mathematician' mostly because it allows to determine when some operations involving categories are 'hierarchically' unequivalent. For instance, we shall soon see that the category of functors between two $\mathbb{U}$-categories is only $\mathbb{V}$-small, and after introducing localization of categories we shall again see that localization of a category always exists only if one moves to a higher universe.

3.3 Functors

It is useful to introduce the notion of a map between different categories, called *functor*:

Definition 3.10. Let $\mathcal{C}, \mathcal{D}$ be two categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

1. A map $F : Ob\,\mathcal{C} \to Ob\,\mathcal{D}$, $X \mapsto FX$,

2. For every $X, Y \in \mathcal{C}$, a map[3] $F : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$, $f \mapsto Ff$.

It is required that

- For every $X \in \mathcal{C}$ $F1_X = 1_{FX}$,

- $F(g \circ f) = Fg \circ Ff$.

Definition 3.11. A functor $F\mathcal{C} \to \mathcal{D}$ is called

- *Faithful* if $\forall X, Y \in \mathcal{C}$, the map $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is injective.

- *Full* if $\forall X, Y \in \mathcal{C}$, the map $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is surjective.

- *Fully faithful* if it is full and faithful.

Definition 3.12. A *subcategory* $\mathcal{C}'$ of $\mathcal{C}$ is a subcollection of objects and morphisms of $\mathcal{C}$, $\mathcal{C}' \subset \mathcal{C}$ and $\mathcal{C}'(X, Y) \subset \mathcal{C}(X, Y)$, which is closed under composition. There is a tautological faithful functor $\mathcal{C}' \hookrightarrow \mathcal{C}$. A subcategory is called *full* if this functor is full.

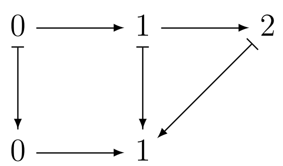
Example 3.13. This example introduces new categories and at the same time shows some first examples of functors

- We can organize $\mathbb{U}$-small categories into a $\mathbb{U}$-category $\mathbf{Cat}_{\mathbb{U}}$ or simply $\mathbf{Cat}$: objects are $\mathbb{U}$-small categories, $\mathbf{Cat}(I, J)$ is the set of all functors from I to J.

- Of particular interest is the subcategory Δ of $\mathbf{Cat}$. Its objects are the categories $[n]$ ($n \in \mathbb{N} = \{0, 1, ...\}$), depicted as

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow n,$$

  So, $Ob\,[n] = \{0, 1, ..., n\}$ and $[n](k, l) = \{*\}$ if $k \leq l$ and is empty otherwise. The composition in $[n]$ is induced uniquely. For $[n], [m] \in \Delta$, we define $\Delta([n], [m])$ to be the set of all functors from $[n]$ to $[m]$. For instance, the diagram



[3]As the reader might note, we repeatedly abuse the notation in this definition.

represents a functor $s : [2] \to [1]$, given on objects by

$$0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$$

The action of s on morphisms is automatic. By examining the definiton, we see that a map of object-sets $f : [n] \to [m]$ can be (uniquely) extended to a functor iff

$$k \leq l \Rightarrow f(k) \leq f(l)$$

for all $k, l \in [n]$.

Example 3.14. Let us mention some examples of functors.

- A functor between $\mathbf{B}G$ and $\mathbf{B}H$ is seen to be a group homomorphism.
- Let X be an object of $\mathcal{C}$. Then one can associate a couple of functors,

  $$h_X : \mathcal{C}^{op} \to \mathbf{Set} \text{ and } h^X : \mathcal{C} \to \mathbf{Set},$$

  in the following way. $h_X(Y) := \mathcal{C}(Y, X)$. On morphisms, h_X sends $f : Z \to Y$ to

  $$f^* : \mathcal{C}(Y, X) \to \mathcal{C}(Z, X),\ (g : Y \to X) \mapsto (g \circ f : Z \to X).$$

  h^X is defined similarly (e.g. $h^X(Y) = \mathcal{C}(X, Y)$).
- The categories from Example 3.5 admit functors to **Set** which send an object to its underlying set[4]. This functor, in essence, forget the 'structure' an object (e.g. a vector space) has, and treat it as a set.
- There is a functor

  $$|-| : \Delta \to \mathbf{Top},$$

  which assigns to $[n]$ the standard n-simplex

  $$|[n]| = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1}_{\geq 0} | \sum_i x_i = 1\}$$

  It sends a morphism $\theta : [n] \to [m]$ to $|\theta| : |[n]| \to |[m]|$, defined as

  $$|\theta|(x_0, ..., x_n) = (y_0, ..., y_m),$$

  where

  $$y_i = \sum_{j \in \theta^{-1}(i)} x_j$$

[4]We leave it to the reader to explain what it means for the category $\mathbf{DGVect}_k$.

if $\theta^{-1}(i)$ is nonempty and $y_i = 0$ otherwise. For example, one can see that an inclusion $[n-1] \to [n]$ sends the $n-1$ simplex to the corresponding face of an n-simplex.

The examples of functors are in abundance and we refer reader to the numerous category theory textbooks if he or she needs more.

3.4 Yoneda lemma

Let $F, G : \mathcal{C} \to \mathcal{D}$ be a couple of functors.

Definition 3.15. A *morphism of functors* from $F \to G$ is a functor $H : \mathcal{C} \times [1] \to \mathcal{D}$, such that $Hi_0 = F$ and $Hi_1 = G$, where $i_k : \mathcal{C} \to \mathcal{C} \times [1]$ $(k = 0, 1)$ is a functor defined on objects as $X \mapsto (X, k)$.

This definition follows the one for a homotopy between maps in topology and is not conventional for category theory textbooxs, so let us recast it in the more familiar form, which we shall conventionally call *natural transformation.* Note that a morphism of functors H sends the diagram[5]

$$\begin{CD} (X,0) @>{(id_X, 0 \to 1)}>> (X,1) \\ @V{(f, id_0)}VV @VV{(f, id_1)}V \\ (Y,0) @>{(id_Y, 0 \to 1)}>> (Y,1) \end{CD}$$

to the diagram

$$\begin{CD} F(X) @>{\alpha_X}>> G(X) \\ @V{F(f)}VV @VV{G(f)}V \\ F(Y) @>{\alpha_Y}>> G(Y) \end{CD}$$

(here $\alpha_X = H((id_X, 0 \to 1))$). Since H is a functor, the resulting diagram commutes. Thus we see that specifying H is essentially the same as specifying a set of arrows $\{\alpha_X : F(X) \to G(X)\}_{X \in \mathcal{C}}$ in $\mathcal{D}$ such that the diagrams like the one above commute.

Definition 3.16. A *natural transformation* $\alpha : F \to G$ is a set of maps $\{\alpha_X : F(X) \to G(X)\}_{X \in \mathcal{C}}$ in the category $\mathcal{D}$ such that for every $f : X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{CD} F(X) @>{\alpha_X}>> G(X) \\ @V{F(f)}VV @VV{G(f)}V \\ F(Y) @>{\alpha_Y}>> G(Y) \end{CD}$$

[5] This is induced in $\mathcal{C} \times [1]$ by a map $f : X \to Y$ in $\mathcal{C}$ and the unique map $0 \to 1$ in $[1]$.

commutes.

Example 3.17. Many natural transformations we are going to work with will be obtained from adjunctions, but there are of course other examples. Consider a functor GL_n : **CRing** $\to$ **Grp** (here **CRing** is the category of commutative rings and **Grp** is the category of groups). This functor sends a ring A to the group $GL_n(A)$ of invertible matrices $n \times n$ with elements in A. This is seen to be a functor (we can always apply a ring morphism $A \to B$ to matrices elementwise).

There is a natural transformation $det : GL_n \to GL_1$ with component arrows

$$det_A : GL_n(A) \to GL_1(A) = A^*, \;\; M \mapsto det\, M.$$

Definition 3.18. Given two categories $\mathcal{C}, \mathcal{D}$ there is a category $Fun(\mathcal{C}, \mathcal{D})$: objects are functors $\mathcal{C} \to \mathcal{D}$, $Hom_{Fun(\mathcal{C},\mathcal{D})}(F, G)$ is the set of all natural transformations from F to G, which we shall denote $Nat(F, G)$.

Remark 3.19. The category $Fun(\mathcal{C}, \mathcal{D})$ is $\mathbb{V}$-small whenever $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{V}$-small. However, when $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{U}$-categories, $Fun(\mathcal{C}, \mathcal{D})$ might not be a $\mathbb{U}$-category. This is only guaranteed when $\mathcal{C}$ is $\mathbb{U}$-small.

A special example of a functor category is the *category of presheaves* on a category $\mathcal{C}$:

Definition 3.20. Given a $\mathbb{U}$-category[6] $\mathcal{C}$, $Pr(\mathcal{C}) := Fun(\mathcal{C}^{op}, \mathbf{Set}_{\mathbb{U}})$. A presheaf of the form

$$h_X : \mathcal{C}^{op} \to \mathbf{Set}_{\mathbb{U}}, \; Y \mapsto \mathcal{C}(Y, X)$$

for $X \in \mathcal{C}$ is called *representable.*

Example 3.21. The *category of simplicial sets* **SSet** is defined to be the category $Fun(\Delta^{op}, \mathbf{Set})$. The representable functor $h_{[n]}$ is also denoted as Δ^n and is called n-simplex.

There are many functors which give examples of simplicial sets.

Example 3.22. The two main examples are:

- There is a functor **Cat** $\to$ **SSet**, which acts as

$$\mathcal{C} \mapsto N\mathcal{C}, \; N\mathcal{C}([n]) = Fun([n], \mathcal{C}).$$

  This functor is called the *nerve functor.*

- There is a functor **Top** $\to$ **SSet**, which acts as

$$X \mapsto Sing(X), \; Sing(X)([n]) = \mathbf{Top}(|\Delta^n|, X).$$

  This functor is called the *singular complex functor.*

[6]Here $\mathbf{Set}_{\mathbb{U}}$ denotes a $\mathbb{U}$-category of all sets in $\mathbb{U}$.

We shall elaborate on these examples later when we start considering simplicial sets in full detail. For now, we turn to one special property of a presheaf category.

Proposition 3.23 (Yoneda Lemma). *For a presheaf $F : \mathcal{C}^{op} \to Set$ and an object $X \in \mathcal{C}$ there exists a bijection*

$$Nat(h_X, F) \cong F(X),$$

which is natural[7] *in X and F.*

Proof. (See e.g. [1] for a detailed exposition) The map

$$Nat(h_X, F) \to F(X)$$

sends $\alpha : h_X \to F$ to $\alpha_X(id_X)$. The inverse map

$$Nat(h_X, F) \leftarrow F(X)$$

sends $t \in F(X)$ to

$$\bar{t}_Y : h_X(Y) \to F(Y),\ \bar{t}_Y(f) = F(f)(t).$$

□

Corollary 3.24. *For $X, Y \in \mathcal{C}$ there is a bijection $\mathcal{C}(X, Y) \cong Nat(h_X, h_Y)$, that is, the functor*

$$h : \mathcal{C} \to Pr(\mathcal{C}),\ X \mapsto h_X$$

is fully faithful.

Proof. $Nat(h_X, h_Y) \cong h_Y(X) = \mathcal{C}(X, Y)$. □

We thus see that the category $\mathcal{C}$ can be, in effect, identified with the full subcategory of $Pr(\mathcal{C})$. We shall see below that the Yoneda lemma allows to describe non-representable objects of $Pr(\mathcal{C})$ as colimits of the objects h_X for $X \in \mathcal{C}$.

3.5 Adjoints and limits

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be a couple of functors.

Definition 3.25. An adjunction $F \dashv G$ is a family of natural isomorphisms

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

between functors $\mathcal{D}(F-, -)$ and $\mathcal{C}(-, G-)$ from $\mathcal{C}^{op} \times \mathcal{D}$ to **Set**. F is called *left adjoint* to G, and G is called *right adjoint* to F.

[7] That is, the bijections form a natural transformation when both sides are treated as functors in the appropriate variable

Proposition 3.26. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor and there are two functors $G_1, G_2 : \mathcal{D} \to \mathcal{C}$ right adjoint to F, then $G_1 \cong G_2$.*

Proof. Excercise on Yoneda Lemma. □

Example 3.27. A great deal of adjoints comes from the forgetful functor. Take e.g. the category $\mathbf{Vect}_k$ and a functor

$$U : \mathbf{Vect}_k \to \mathbf{Set}$$

which sends a vector space V to the set UV of vectors of V. This functor has a left adjoint

$$F : \mathbf{Set} \to \mathbf{Vect}_k$$

. It sends a set S to the vector set $\bigoplus_S k$, that is, the free vector space with basis S. One can see that this defines a functor which also is a left adjoint to U. This is a common example of the "Forget Structure — Form Free Object" situation.

There is a special class of adjoint functors which is of great importance.

Consider a category $\mathcal{C}$ and a small category I. We may employ the following terminology: a functor $D : I \to \mathcal{C}$ will be called an I-*diagram*, or simply *diagram* in $\mathcal{C}$. There is a functor

$$\Delta_I : \mathcal{C} \to Fun(I, \mathcal{C})$$

which sends an object X to the functor

$$i \xrightarrow{u} j \quad \mapsto \quad X \xrightarrow{id_X} X$$

That is, the functor Δ_I sends an object X to the constant diagram at this object.

Definition 3.28. A *colimit functor* $\varinjlim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$ is a left adjoint of the functor Δ_I. A *limit functor* $\varprojlim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$ is a right adjoint of the functor Δ_I[8].

Limit and colimit functors need not exist for a category $\mathcal{C}$. It is also often the case that they exist only for special I and/or are only defined at special $D \in Fun(I, \mathcal{C})$.

Given a diagram D, a limit of this diagram is an object $\varprojlim D$. Take the adjunction map,

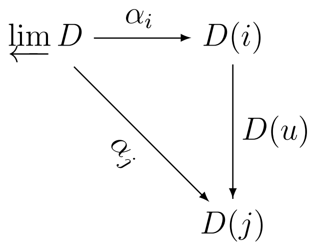
$$Nat(\Delta_I T, D) \cong \mathcal{C}(T, \varprojlim D),$$

and plug $T = \varprojlim D$. Then there is $id_{\varprojlim D} \in \mathcal{C}(\varprojlim D, \varprojlim D)$, which gives a map $\alpha : \Delta_I \varprojlim D \to D$. This natural transformation is, for every object i of I, a family of maps

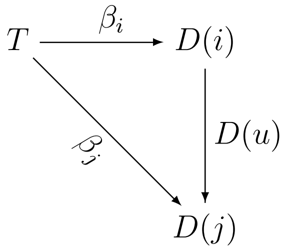
$$\alpha_i : \varprojlim D \to D(i)$$

[8]Most of the time the index I in $\varinjlim_I$ and $\varprojlim_I$ will be suppressed.

in $\mathcal{C}$, such that for any morphism $u : i \to j$ in I the diagram



commutes. Moreover, suppose $T \in \mathcal{C}$ is equipped with a morphism $\beta : \Delta_I T \to D$, that is, a family of maps $\beta_i : T \to D(i)$ in $\mathcal{C}$, such that for any morphism $u : i \to j$ in I the diagram



commutes. Then the adjunction isomorphism tells us that there is a unique map $f_T : T \to \varprojlim D$ such that $\beta_i = \alpha_i \circ f_T$ for every $i \in I$. This is the universal property which forms the 'traditional' definition of a limit of a diagram. We also see that for a diagram D, any two limits, if they exist, are isomorphic by a *unique* isomorphism.

We leave it to the reader to determine the universal property of a colimit, which is a limit in $\mathcal{C}^{op}$, and turn to the examples.

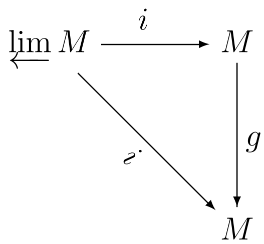
Example 3.29. (Limits and colimits of selected diagrams)

IN-FIN $I = \emptyset$, that is, the empty category. Then $Fun(I, \mathcal{C}) = *$, that is, the category with one object and one morphism, so colimit and limit functors select objects 0 and 1 in $\mathcal{C}$ respectively. The object 0 has the property that, for every $X \in \mathcal{C}$, there is a unique map $0 \to X$. The object 1 has the property that, for every $X \in \mathcal{C}$, there is a unique map from $X \to 1$. 0 and 1 are known as *an initial* and *a final* object of $\mathcal{C}$ respectively. Note that $I = \emptyset$ is the initial object of **Cat** and $J = *$ is a final object of **Cat**.

INV It is not interesting to consider $I = *$. Rather, take $I = \mathbf{B}G$, where G is a group. Let us consider the case $\mathcal{C} = \mathbf{Set}$. Then $Fun(\mathbf{B}G, \mathbf{Set}) = G - \mathbf{Set}$, the category of sets with left G-action. Indeed, a functor $\mathbf{B}G \to \mathbf{Set}$ sends the unique object $*$ of $\mathbf{B}G$ to a set M, and on morphisms it gives a map of sets $G \to \mathrm{End}(M) = \mathbf{Set}(M, M)$.

A limit of $M \in G - \mathbf{Set}$ would be an object $\varprojlim M$ of **Set** with a map of sets $i : \varprojlim M \to M$

such that for every $g \in G$ considered as a map $g : M \to M$ the diagram



commutes. Moreover, it should satisfy the universal property of a limit. In light of this, take

$$\varprojlim M = M^G = \{m \in M \,|\forall\, g \in G\ gm = m\}.$$

This is the set of *invariants* of M. It is easily checked that this set satisfies the universal property of a limit.

Dually,

$$\varinjlim M = M_G = M/\sim,$$

where the equivalence relation is

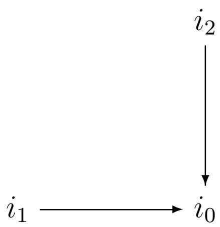
$$m \sim m' \Leftrightarrow \exists g \in G : m = gm'.$$

This is the set of *coinvariants* of M.

PROD When $I = \bullet\ \bullet$, that is, the category with 2 objects and 2 morphisms, an I-diagram in $\mathcal{C}$ is just a couple of objects X and Y in $\mathcal{C}$. In this case, a limit of the diagram with values X, Y is denoted $X \times Y$ or $X \prod Y$ and is called a *product* of X and Y, and a colimit of such diagram is denoted $X \coprod Y$ and is called a *coproduct* of X and Y. The reader might see that for $\mathcal{C} = \mathbf{Set}$, $X \prod Y$ is the cartesian product of sets and $X \coprod Y$ is the disjoint union of sets. However, the reader also might check that for e.g. $\mathbf{Vect}_k$, *both* products and coproducts are given by the direct sum[9] of vector spaces.

PROD' More generally, one can consider products and coproducts of arbitrary number of objects. This can be done by taking I to be the category with a $\mathbb{U}$-set of objects and no non-identity morphisms. The (co)limit over a diagram $I \to \mathcal{C}$ is called a *small (co)product* of the corresponding set of objects in $\mathcal{C}$.

FIB There is a diagram category I_{Π}:



[9] It is crucial here that I has only a finite set of objects.

A functor $I_\Pi \to \mathcal{C}$ gives a diagram

$$\begin{array}{ccc} & & X_2 \\ & & \big\downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow{f_1} & X_0 \end{array}$$

The limit of this diagram,

$$\begin{array}{ccc} X_1 \prod\limits_{X_0} X_2 & \longrightarrow & X_2 \\ \big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow{f_1} & X_0 \end{array}$$

is called the *fibered product* of maps $f_i : X_i \to X_0$, or the *pullback* of f_1 along f_2. One checks explicitly that in many familiar categories, e.g. **Set**, the pullback can be chosen equal to a certain subset of the product $X_1 \times X_2$:

$$X_1 \prod_{X_0} X_2 = \{(x_1, x_2) \in X_1 \times X_2 \,|\, f_1(x_1) = f_2(x_2)\,\}.$$

One can try the same construction in other categories, e.g. **Top**, $\mathbf{Mod}_A$ and **Cat**.

COFIB Dually, the colimit of a functor $I_\Pi^{op} \to \mathcal{C}$ leads us to the following diagram:

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\big\downarrow & & \big\downarrow \\ X_2 & \longrightarrow & X_1 \coprod\limits_{X_0} X_2 \end{array}$$

The colimit $X_1 \coprod_{X_0} X_2$ of this diagram is called the *cofibered product* of f_1 and f_2, or the *pushout* of f_1 along f_2. In many categories one can compute cofibered products by taking quotients of coproducts. For instance, in **Set** one might define the equivalence relation on $X_1 \coprod X_2$ which glues $x_1 \in X_1$ with $x_2 \in X_2$ iff there exist an element $x_0 \in X_0$ such that $f_1(x_0) = x_1$, $f_2(x_0) = x_2$. However, usually the procedure of taking quotients may be quite complicated, as the reader might see on the example of $\mathcal{C} = \mathbf{Cat}$.

EQ Lastly, one can consider a diagram of the form

$$i_1 \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} i_2$$

The limit of the corresponding functor,

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y,$$

is called an *equalizer* of f and g. Equalizers are familiar for many in categories like $\mathbf{Vect}_k$ or $\mathbf{Mod}_A$: there one observes that the kernel of a map $f : M \to N$ has the universal property of an equalizer of f and the zero map $0 : M \to N$.

Taking colimit of this diagram gives the dual notion of *coequalizer*.

Proposition 3.30. *ADJOINT FUNCTORS PRESERVE LIMITS (WILL BE ADDED LATER)*

Proposition 3.31. *For a category $\mathcal{C}$, admitting small (co)limits is the same as admitting small (co)products and (co)equalisers.*

Before we continue, let us introduce the notation. For a category $\mathcal{C}$, define

$$\operatorname{Mor} \mathcal{C} := \coprod_{X,Y \in \mathcal{C}} \mathcal{C}(X, Y).$$

This is the *set of all morphisms* of $\mathcal{C}$. (in general only $\mathbb{V}$-small). There are two maps $s, t : \operatorname{Mor} \mathcal{C} \to \operatorname{Ob} \mathcal{C}$. s sends a morphism $f : X \to Y$ to X and t maps f to Y.

Proof. Let us assume that small products and equalizers exist. Then, for a diagram $D : I \to \mathcal{C}$ we construct the product

$$A = \prod_{i \in \operatorname{Ob} I} D(i).$$

For every $\alpha : i \to j \in \operatorname{Mor} I$ there are two maps from A to $D(j) = D(t(\alpha))$. The first one is given by the projection $\pi_{t(\alpha)} : A \to D(t(\alpha))$. This map exists since A is the product of all objects in the diagram, and products come equipped with projections. The second one is given by the composition $D(\alpha) \circ \pi_{s(\alpha)} : A \to D(s(\alpha)) \to D(t(\alpha))$. Since these two maps are present for every $\alpha \in \operatorname{Mor} I$ we observe that if we form the product

$$B = \prod_{\alpha \in \operatorname{Mor} I} D(t(\alpha))$$

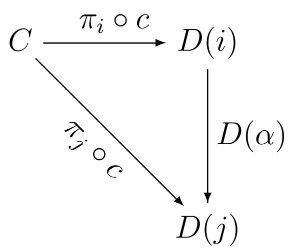
then by the universal property for this product there are two maps

$$A \underset{\prod_{\alpha \in \operatorname{Mor} I} D(\alpha) \circ \pi_{s(\alpha)}}{\overset{\prod_{\alpha \in \operatorname{Mor} I} \pi_{t(\alpha)}}{\rightrightarrows}} B$$

Take C to be the equalizer of these two maps. Let $c : C \to A$ be the universal map for which we know that

$$\left(\prod_{\alpha \in \mathrm{Mor}\, I} \pi_{t(\alpha)}\right) \circ c = \left(\prod_{\alpha \in \mathrm{Mor}\, I} D(\alpha) \circ \pi_{s(\alpha)}\right) \circ c.$$

- For every $i \in I$ C comes equipped with natural maps to $D(i)$ given by the composition $\pi_i \circ c$. The diagram

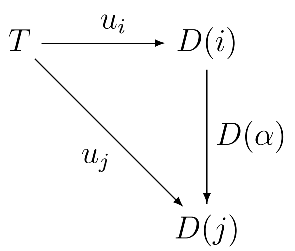


  morever commutes due to the fact that

$$\pi_j \circ \left(\prod_{\alpha \in \mathrm{Mor}\, I} \pi_{t(\alpha)}\right) \circ c = \pi_j \circ \left(\prod_{\alpha \in \mathrm{Mor}\, I} D(\alpha) \circ \pi_{s(\alpha)}\right) \circ c$$

  and one checks that $\pi_j \circ \left(\prod_{\alpha \in \mathrm{Mor}\, I} \pi_{t(\alpha)}\right) = \pi_j$ and $\pi_j \circ \left(\prod_{\alpha \in \mathrm{Mor}\, I} D(\alpha) \circ \pi_{s(\alpha)}\right) = D(\alpha) \circ \pi_i$.

- If T is an object with a set of maps $u_i : T \to D(i)$ such that



  commutes for every $\alpha : i \to j$ then $\prod_{i \in \mathrm{Ob}\, I} u_i : T \to A$ is a morphism such that

$$\left(\prod_{\alpha \in \mathrm{Mor}\, I} \pi_{t(\alpha)}\right) \circ \left(\prod_{i \in \mathrm{Ob}\, I} u_i\right) = \left(\prod_{\alpha \in \mathrm{Mor}\, I} D(\alpha) \circ \pi_{s(\alpha)}\right) \circ \left(\prod_{i \in \mathrm{Ob}\, I} u_i\right).$$

  Because of this there is a unique map $f : T \to C$ such that everything, what needs to commute, commutes.

The proof for colimits is automatic after the replacement of $\mathcal{C}$ by $\mathcal{C}^{op}$. □

Corollary 3.32. *Small limits and colimits exist in* **Set**, **Top**, **Mod**, **DGMod**.

Proof. Excercise on applying Proposition 3.31 and recognizing (co)products and (co)equalizers in the corresponding categories. □

Lemma 3.33. *If $\mathcal{D}$ is a category such that small (co)limits exist in $\mathcal{D}$, then for any $\mathbb{U}$-category $\mathcal{C}$, (co)limits exist in the category $Fun(\mathcal{C}, \mathcal{D})$.*

Proof. Let $D : I \to Fun(\mathcal{C}, \mathcal{D})$ be a diagram. For every object X in $\mathcal{C}$ there is a functor

$$ev_X : Fun(\mathcal{C}, \mathcal{D}) \to \mathcal{D},\ F \mapsto F(X).$$

This gives a diagram $D_X = ev_X \circ D : I \to \mathcal{D}$. Using this, define the functor $\varprojlim D : \mathcal{C} \to \mathcal{D}$: on the objects

$$(\varprojlim D)(X) := \varprojlim D_X.$$

One checks that this can be extended to morphisms in $\mathcal{C}$ and then verifies the required universal property of a limit. For colimits, the proof is similar. □

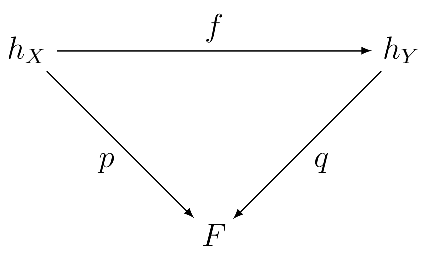
Corollary 3.34. *For a category $\mathcal{C}$, $Pr(\mathcal{C})$ is complete (that is, admits small limits) and cocomplete (that is, admits small colimits). In particular, this is true for* **SSet**.

3.6 Geometric realization

We have seen that there are functors N : **Cat** $\to$ **SSet** and $Sing$: **Top** $\to$ **SSet**, and both of them are defined as the composition of the Hom-functor and some functor $\Delta \to$ **Cat** or $\Delta \to$ **Top**. In this section we construct left adjoints to these functors.

Definition 3.35. Given a category $\mathcal{C}$ and a presheaf $F \in Pr(\mathcal{C})$ the *category of elements* of F is denoted by $\mathcal{C}_F$ and consists of the following

- The objects of $\mathcal{C}_F$ are morphisms $p : h_X \to F$, that is, elements of $F(X)$. We shall use the same letter to denote $p \in F(X)$ and $p : h_X \to F$.
- An element of $\mathcal{C}_F(p, q)$ is a commutative triangle in $Pr(\mathcal{C})$:



  If we think about p and q as elements of $F(X)$ and $F(Y)$, $\mathcal{C}_F(p, q)$ consists of all morphisms $f : X \to Y$ such that $F(f)(q) = p$.

There is a functor $\pi_F : \mathcal{C}_F \to \mathcal{C}$ which maps $p : h_X \to F$ to X. This is well-defined due to the Yoneda lemma.

Theorem 3.36. *Let $\mathcal{C}$ be a category, $\mathcal{E}$ be a cocomplete category and $A : \mathcal{C} \to \mathcal{E}$ be a functor. Then the functor $R : \mathcal{E} \to Pr(\mathcal{C})$ which maps an object E to the functor*

$$X \mapsto \mathcal{E}(A(X), E)$$

has a left adjoint $L\mathcal{E} \to Pr(\mathcal{C})$, which maps F to the colimit of the functor $A \circ \pi_F : \mathcal{C}_F \to \mathcal{E}$. Moreover, the diagram

$$\begin{array}{ccc} Pr(\mathcal{C}) & \xrightarrow{L} & \mathcal{E} \\ {\scriptstyle h}\uparrow & \nearrow{\scriptstyle A} & \\ \mathcal{C} & & \end{array}$$

where h is the Yoneda embedding functor, commutes up to isomorphism.

Proof. A morphism of functors

$$\tau : F \to R(E)$$

gives, for every X in $\mathcal{C}$, a morphism $\tau_X : F(X) \to R(E)(X) = \mathcal{E}(A(X), E)$. That is, for every $p : h_X \to F$ we are given a map $\tau_X(p) : A(X) \to E$. If $f : p \to q$ is a morphism in $\mathcal{C}_F$, then, since τ is a natural transformation, we have $\tau_X(p) = \tau_Y(q) \circ A(f)$. This is a direct consequence of the fact that diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{\tau_X} & \mathcal{E}(A(Y), E) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle A(f)^*} \\ F(X) & \xrightarrow{\tau_Y} & \mathcal{E}(A(X), E) \end{array}$$

commutes and that $F(f)(q) = p$. All this means that we get a morphism of diagrams

$$\tilde{\tau} : A \circ \pi_F \to \Delta_{\mathcal{C}_F} E$$

and moreover giving a morphism $A \circ \pi_F \to \Delta_{\mathcal{C}_F} E$ is equivalent to specifying a morphism $F \to R(E)$. But the morphism $\tilde{\tau}$ is the same as

$$\bar{\tau} : \varinjlim A \circ \pi_F \to E.$$

All constructions are seen to be natural in E and F and so give us the desired adjunction.

If one takes $F = h_X$ then, due to the fact that the category $\mathcal{C}_{h_X}$ has the terminal object $id_X : h_X \to h_X$ we easily check that $\varinjlim A\pi_{h_X} \cong A(X)$. This gives us the desired commutativity of $L \circ h$ and A. $\square$

Corollary 3.37. *Every presheaf F is isomorphic to the colimit of the diagram of representable functors indexed by $\mathcal{C}_F$.*

Proof. In Theorem 3.36 take $\mathcal{E}$ equal to $Pr(\mathcal{C})$ and $A = h$. Then use the fact that $R(F)$ is canonically isomorphic F and thus R is isomorphic to $id_{Pr(\mathcal{C})}$. And $id_{Pr(\mathcal{C})}$ is both its own left and right adjoint. □

Corollary 3.38. *There is a functor* $|-| : \mathbf{SSet} \to \mathbf{Top}$ *which is left adjoint to* $Sing : \mathbf{Top} \to \mathbf{SSet}$ *and sends the representable functors* $\Delta^n = \Delta(-, [n])$ *to the realization of* $[n]$. *It is called the* geometric realization *functor.*

There is also a functor $C : \mathbf{SSet} \to \mathbf{Cat}$ *which is left adjoint to* $N : \mathbf{Cat} \to \mathbf{SSet}$ *and sends the representable functors* $\Delta^n = \Delta(-, [n])$ *to* $[n]$. *It is called the* categorical realization *functor.*

Remark 3.39. Let us elaborate a bit on the example of **SSet**. We learnt from Proposition 3.31 that a colimit may be specified as a coequalizer between two small coproducts. That is, if $X \in \mathbf{SSet}$ and Δ_X is the associated category of elements, then the geometric realization of X can be chosen as a colimit of the diagram[10]

$$\coprod_{\alpha \in \mathrm{Mor}\,\Delta_X} |\Delta^{s(\alpha)}| \underset{\coprod_{\alpha \in \mathrm{Mor}\,\Delta_X} i_{t(\alpha)} \circ \alpha}{\overset{\coprod_{\alpha \in \mathrm{Mor}\,\Delta_X} i_{s(\alpha)}}{\rightrightarrows}} \coprod_{\Delta^n \in \Delta_X} |\Delta^n|.$$

Decyphering it a bit, we see that

$$\coprod_{\Delta^n \in \Delta_X} |\Delta^n| = \coprod_{n \in \mathbb{N}} X([n]) \times |\Delta^n|$$

where the product is taken in **Top** and $X([n])$ is given discrete topology. Moreover, examining the colimit diagram (and keeping in mind Definition 3.35) it is seen that in order to coequalize the two maps we have to quotient the coproduct $\coprod_{n \in \mathbb{N}} X([n]) \times |\Delta^n|$ by the relations of the form

$$(X(\alpha)(q), x) \sim (q, |\alpha|(x)).$$

Here $\alpha : p \to q$ is a map in ΔX, that is, a morphism $\alpha : [n] \to [m]$ such that it $p \in X([n])$ is equal to $X(\alpha)(q)$ for $q \in X([m])$. Also, $x \in |\Delta^n|$ is a point of the realization of $[n]$. These relations exist for arbitrary $n \in \mathbb{N}, q$ and x; they appear right after one disassembles the coproducts.

Putting all this together, $|X| = \coprod_{n \in \mathbb{N}} X([n]) \times |\Delta^n| / \sim$, which is the classical expression for geometric realization functor. The advantage of our approach is that functoriality left-adjointness of this expression is readily manifest, yet one might have issues understanding what the realization looks like.

[10]The notation is a bit ugly and will be explained in a later version.

The remark applies to the category of simplicial sets as well. That is, in **SSet** we have

$$X([k]) = \coprod_{n\in\mathbb{N}} X([n]) \times \Delta^n([k])/\sim$$

where the equivalence relation has the form

$$(X(\alpha)(q), s) \ \sim\ (q, \alpha(s)).$$

Here, again, $\alpha : p \to q$ is a map in ΔX, that is, a morphism[11] $\alpha : [n] \to [m]$ such that it $p \in X([n])$ is equal to $X(\alpha)(q)$ for $q \in X([m])$. Also, $x \in \Delta^n([k])$ is a k-simplex of Δ^n.

This remark may be used to observe the following property of simplicial sets. Before we state it, let us give a

Definition 3.40. A n-simplex x of a simplicial set X is called *degenerate* if there exists a surjective map $\varphi : [n] \to [k]$ in Δ such that $x = X(\varphi)(y)$ for some k-simplex y. Otherwise x is called *non-degenerate.*

Proposition 3.41. *For a simplicial set X we have $X \cong \varinjlim_{\Delta_X^{nd}} \Delta^n$ and $|X| \cong \varinjlim_{\Delta_X^{nd}} |\Delta^n|$, where Δ_X^{nd} is a full subcategory of Δ_X whose objects $p : \Delta^n \to X$ are nondegenerate simplices.*

Proof. Let $p = X(\varphi)(q)$ for a surjective map $\varphi : [n] \to [k]$. We may assume that q is non-degenerate though this is not necessary. The relation

$$(p, s) \ \sim\ (q, \varphi(s)).$$

shows us the following. Any $s \in \Delta^n([m])$ should be identified with its image in $\Delta^k([m])$. Since φ is surjective we observe that, when we quotient by the equivalence relation, all m-simplices of p become identified with some m-simplices of q. A similar proof applies to the geometric realization. □

Remark 3.42. Let $f : X \to Y$ be a map of simplicial sets. Then we observe that there is a natural functor, $\Delta_f : \Delta_X \to \Delta_Y$ which takes $\Delta^n \to X$ to $\Delta^n \to X \xrightarrow{f} Y$. However, the construction Δ_X^{nd} is not functorial in X.

4 Localisation of categories

We now turn to the formal description of categories objects of which may be 'weakly equivalent' via elements of some distinguished class of morphisms. Surprisingly, there is a lot of words one can say about such categories and functors between them at the formal level and, unsurprisingly, not many introduced notions can be proven to exist. This will be solved by the machinery of model categories, which we shall introduce in the next section.

[11]So, given the fact that $s \in \Delta^n[k]$ is a morphism $s : [k] \to [n]$, the expression $\alpha(s)$ means $\alpha \circ s : [k] \to [m]$.

4.1 Localisers

Definition 4.1. A *localiser* is a category $\mathscr{M}$ together with a set of maps $\mathscr{W} \subset \mathrm{Mor}\,\mathscr{M}$, called the set of *weak equivalences* of $\mathscr{M}$. In this case we might also say that $\mathscr{M}$ is a *category with weak equivalences* $\mathscr{W}$. A *morphism of localisers* $(\mathscr{M},\mathscr{W}) \to (\mathscr{N},\mathscr{W}')$ is a functor $F : \mathscr{M} \to \mathscr{N}$ such that $F(\mathscr{W}) \subseteq \mathscr{W}'$. Such functors are also called *homotopically meaningful functors*

Stated this way, the notion of a homotopically meaningful functor is usually obtained not from 'working mathematics' but from formal examples.

Example 4.2. The formalities to keep in mind now follow:

ISO Any category $\mathcal{C}$ can be transformed into a localiser $(\mathcal{C}, Iso_{\mathcal{C}})$. Here $Iso_{\mathcal{C}}$ is the set of all isomorphisms of $\mathcal{C}$. Any functor sends isomorphisms to isomorphisms, so we can embed categories into localisers this way. Also, if $(\mathscr{M},\mathscr{W})$ is a localiser, we say that a functor $F : \mathscr{M} \to \mathcal{C}$ is homotopically meaningful if it is a morphism of localisers $(\mathscr{M},\mathscr{W}) \to (\mathcal{C}, Iso_{\mathcal{C}})$.

DIA Let $(\mathscr{M},\mathscr{W})$ be a localiser, $u : I \to J$ a functor between (small) categories. Then the category $Fun(I,\mathscr{M}) =: \mathscr{M}^I$ can be given the structure of a localizer with the set of weak eqiuvalences $\mathscr{W}_I$: a morphism $\alpha : F \to G$ in $\mathscr{M}^I$ is in $\mathscr{W}_I$ iff for all $i \in I$ $\alpha(i) : F(i) \to G(i)$ is in $\mathscr{W}$. Moreover, the functor u induces a functor

$$u^* : \mathscr{M}^J \to \mathscr{M}^I,\ F \mapsto F \circ u.$$

It is checked that u^* is a morphism of localisers $(\mathscr{M}^J, \mathscr{W}_J) \to (\mathscr{M}^I, \mathscr{W}_I)$.

We shall see later that 'natural' examples of functors between localisers are not usually homotopically meaningful 'on the nose'. Before that, let us introduce 'the working mathematician's' examples of localisers themselves.

Example 4.3.

TOP The category of topological spaces **Top** is usually equipped with the class $\mathscr{W}_{\mathbf{Top}} = Weq$, that is, with the maps $X \to Y$ which are weak homotopy equivalences of topological spaces.

SSET The category **SSet** of simplicial sets has a geometric realization functor $|-| : \mathbf{SSet} \to \mathbf{Top}$. We shall make this functor into a morphism of localisers. Define $\mathscr{W}_{\mathbf{SSet}}$ in $Arr(\mathbf{SSet})$ to be the set of all maps $\alpha : S \to S'$ such that $|\alpha| : |S| \to |S'|$ is a weak homotopy equivalence.

CAT The category of small categories **Cat** is equipped with the class $\mathscr{W}_{\mathbf{Cat}}$, which consists of all *equivalences* of small categories.

An *equivalence* $F : \mathcal{C} \to \mathcal{D}$ is a functor such that there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $F \circ G \cong id_{\mathcal{D}}$ and $G \circ F \cong id_{\mathcal{C}}$. It is a theorem that in this case F is fully faithful, and also for any object Y of $\mathcal{D}$ there exists an object X in $\mathcal{C}$, such that $Y \cong F(X)$. This last property means that F is *essentially surjective*. One can prove that a fully faithful and essentially surjective functor is an equivalence of categories.

CAT' A *Morita equivalence* of categories is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F^* : Pr(\mathcal{D}) \to Pr(\mathcal{C})$ is an equivalence of categories. Define $\mathscr{ME} \subset Arr(\mathbf{Cat})$ to be the set of all Morita equivalences. One observes that $id_{\mathbf{Cat}} : (\mathbf{Cat}, \mathscr{W}_{\mathbf{Cat}}) \to (\mathbf{Cat}, \mathscr{ME})$ is a morphism of localisers, but not every Morita equivalence is an equivalence of categories.

CAT" Yet another example is provided by considering the functor $N : \mathbf{Cat} \to \mathbf{SSet}$. Define $\mathscr{HE}$ to be the set of *Thomason equivalences* of categories, that is, functors $F : \mathcal{C} \to \mathcal{D}$ such that $|NF| : |N\mathcal{C}| \to |N\mathcal{D}|$ is a weak homotopy equivalence[12]

DG Take a ring A and the category $\mathbf{DGMod}_A$ from the Example 3.5. Then there is a $\mathbb{Z}$-indexed series of functors $H^n : \mathbf{DGMod}_A \to \mathbf{Mod}_A$. On objects, H^n takes $M^\bullet$,

$$\dots \xrightarrow{d_M^{i-2}} M^{i-1} \xrightarrow{d_M^{i-1}} M^i \xrightarrow{d_M^i} M^{i+1} \xrightarrow{d_M^{i+1}} \dots,$$

to $H^n(M^\bullet) = kerd_M^i / imd_M^{i-1}$. It can be checked that a morphism of complexes $f : M^\bullet \to N^\bullet$ gives a morphism $H^n(f) : H^n(M^\bullet) \to H^n(N^\bullet)$ (excercise). We define $\mathscr{W}_{\mathbf{DGMod}_A}$ to be the set of all morphisms f such that for every $n \in \mathbb{Z}$ $H^n(f)$ is an isomorphism. Such morphisms are called *quasiisomorphisms* of complexes.

4.2 Localisation

Given a localiser $(\mathscr{M}, \mathscr{W})$ we would like not to distinguish objects connected by (a chain of) morphisms from $\mathscr{W}$. We shall now formally define what this means.

Definition 4.4. Let $(\mathscr{M}, \mathscr{W})$ be a localiser. A localisation of $\mathscr{M}$ along $\mathscr{W}$ is a category $\mathscr{M}[\mathscr{W}^{-1}]$ with a functor $p : \mathscr{M} \to \mathscr{M}[\mathscr{W}^{-1}]$ such that

1. p is a morphism of localisers $(\mathscr{M}, \mathscr{W}) \to (\mathscr{M}[\mathscr{W}^{-1}], Iso_{\mathscr{M}[\mathscr{W}^{-1}]})$,
2. For any category $\mathcal{C}$ and a morphism of localisers $F : (\mathscr{M}, \mathscr{W}) \to (\mathcal{C}, Iso_{\mathcal{C}})$ there is a unique functor $\bar{F} : \mathscr{M}[\mathscr{W}^{-1}] \to \mathcal{C}$ such that $F = \bar{F} \circ p$.

Remark 4.5. The definition we used above is 'evil' in the sense that it requires the functors to be equal on the nose rather than be isomorphic. We shall see below that in most examples we can get away with this definition. The definition, however, may be relaxed so to avoid the use of functor equalities.

Proposition 4.6. *For a localiser $(\mathscr{M}, \mathscr{W})$ with $\mathscr{M}$ a $\mathbb{U}$-category, the localisation always exists (assuming that the resulting category $\mathscr{M}[\mathscr{W}^{-1}]$ is small with respect to a higher universe).*

[12]It can be shown that geometric realization of simplicial sets preserves products, and from this one can see that an equivalence of categories is a Thomason equivalence.

Proof.

PUSH The easiest way to prove it is as follows. Observe that a morphism $s : X \to Y$ in $\mathscr{W}$ gives a functor $s : [1] \to \mathscr{M}$: $0 \mapsto X$, $1 \mapsto Y$ and the unique morphism $0 \to 1$ goes to s. There is also a natural functor $[1] \to [1]_{iso}$, where $[1]_{iso}$ is the category with objects $0, 1$ and a unique non-identity isomorphism between them. Overall a morphism $s \in \mathscr{W}$ gives us a diagram of categories

$$\begin{array}{ccc} [1] & \xrightarrow{s} & \mathscr{M} \\ \downarrow & & \\ [1]_{iso} & & \end{array}$$

One observes that the pushout $\mathscr{M} \coprod_{[1]} [1]_{iso}$ is in fact the localisation $\mathscr{M}[s^{-1}]$ of $\mathscr{M}$ with respect to the set of maps $\{s\}$. Considering different $s \in \mathscr{W}$ gives us a diagram

$$\begin{array}{ccc} \coprod_{s \in \mathscr{W}} [1] & \xrightarrow{\coprod s} & \mathscr{M} \\ \downarrow & & \\ \coprod_{s \in \mathscr{W}} [1]_{iso} & & \end{array}$$

The colimit of this diagram is seen to satisfy the universal property of a localisation.

ZAG We might do something more explicit. For $X, Y \in \mathscr{M}$ define $ZZ(X, Y)$ to be the set of all zigzags of morphisms between X and Y of the form

$$X \to T_0 \leftarrow T_1 \cdots \to T_n \leftarrow Y$$

Here the number of morphisms between X and Y can be any element of $\mathbb{N}$ (possibly 0). The morphisms in the direction of Y are arbitrary, the morphisms in the direction of X must belong to $\mathscr{W}$. We also define a relation $\sim$ on $ZZ(X, Y)$, which is the minimal equivalence relation containing the following elementary relations:

$$T_1 \xrightarrow{u} T_2 \xrightarrow{v} T_3 \quad \sim \quad T_1 \xrightarrow{v \circ u} T_3,$$

$$T_1 \xrightarrow{s} T_2 \xleftarrow{s} T_1 \quad \sim \quad T_1 \xrightarrow{id_{T_1}} T_1,$$

$$T_2 \xleftarrow{s} T_1 \xrightarrow{s} T_2 \quad \sim \quad T_2 \xrightarrow{id_{T_2}} T_2,$$

$$T \xrightarrow{id_T} T \quad \sim \quad T,$$

where $u, v \in \mathrm{Mor}(\mathscr{M})$ and $s \in \mathscr{W}$, and in the last line T means the zigzag of zero length. Define then $\mathrm{Ob}\,\mathscr{M}[\mathscr{W}^{-1}] = \mathrm{Ob}\,\mathscr{M}$ and $\mathscr{M}[\mathscr{W}^{-1}](X,Y) = ZZ(X,Y)/\ $. There is a natural composition inherited from the composition in $\mathscr{M}$, and also a functor $\mathscr{M} \to \mathscr{M}[\mathscr{W}^{-1}]$ which sends a morphism $f : X \to Y$ to the class of a unit length zigzag given by f. The reader verifies that this construction gives a localisation functor.

In both cases we cannot guarantee that the resulting category will be a $\mathbb{U}$-category. Taking coproducts requires to move to a higher universe, and the zigzag construction shows us that the set $ZZ(X,Y)$ may have cardinality equal to that of $\mathrm{Ob}\,\mathscr{M}$. □

4.3 Derived functors

Not all functors between localizers are homotopically meaningful.

Example 4.7. Take the category $\mathbf{DGMod}_{\mathbb{Z}}$, that is, the category of complexes of abelian groups. Take a group $\mathbb{Z}/n\mathbb{Z}$ and consider a functor

$$- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} : \mathbf{DGMod}_{\mathbb{Z}} \to \mathbf{DGMod}_{\mathbb{Z}},\ A \mapsto A \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}.$$

This functor does not take quasiisomorphisms to quasiisomorphisms. For example, a sequence of group morphisms

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

when viewed as a complex has zero[13] cohomology groups, and so it gives rise to a quasiisomorphism from

$$\ldots \to \mathbb{Z} \to \mathbb{Z} \to \ldots$$

to

$$\ldots \to 0 \to \mathbb{Z}/n\mathbb{Z} \to \ldots$$

But after tensoring the complexes with $\mathbb{Z}/n\mathbb{Z}$ we get a map from

$$\ldots \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to \ldots$$

with zero differentials to

$$\ldots \to 0 \to \mathbb{Z}/n\mathbb{Z} \to \ldots$$

Since the differentials in the first complex are zero, one observes that cohomology groups are different for these two complexes.

The example of functor above is however interesting and natural, so we might hope that a good deal of functors leaves at least some information at homotopical level. Non-invertible natural transformations allow us to make one step further in this direction.

[13] Here the map $\mathbb{Z} \to \mathbb{Z}$ is given by $x \mapsto nx$.

Remark 4.8. Let $F : \mathcal{C} \to \mathcal{D}$ and $G, H : \mathcal{D} \to \mathcal{E}$ be three functors and $\alpha : G \to H$ be a natural transformation. Then there is a natural transformation $\alpha * F : G \circ F \to H \circ F$ which is defined as

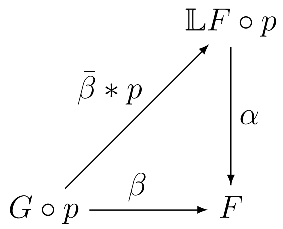
$$G \circ F(X) \overset{\alpha_{F(X)}}{\to} H \circ F(X).$$

This family of arrows is readily verified to be a natural transformation and provides us an example of horizontal composition of natural transformations[14].

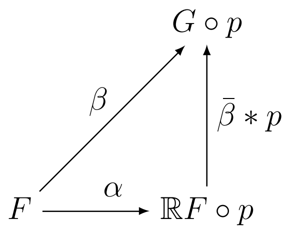
We can dualize the situation and consider three functors $F : \mathcal{D} \to \mathcal{E}$ and $G, H : \mathcal{C} \to \mathcal{D}$, and for every $\alpha : G \to H$ we can define $F * \alpha : F \circ G \to F \circ H$.

Definition 4.9. Let $(\mathscr{M}, \mathscr{W})$ be a localiser, $p : \mathscr{M} \to \mathscr{M}[\mathscr{W}^{-1}]$ be a localisation functor and $F : \mathscr{M} \to \mathcal{D}$ be a functor.

$\mathbb{L}$ A *left derived functor* of F is a pair $(\mathbb{L}F, \alpha)$ consisting of a functor $\mathbb{L}F : \mathscr{M}[\mathscr{W}^{-1}] \to \mathcal{D}$ and a natural transformation $\alpha : \mathbb{L}F \circ p \to F$, which furthermore is universal in the following sense. For every pair (G, β) of $G : \mathscr{M}[\mathscr{W}^{-1}] \to \mathcal{D}$ and $\beta : G \circ p \to F$ there is a unique natural transformation $\bar{\beta} : G \to \mathbb{L}F$ such that $\beta = \alpha \circ (\bar{\beta} * p)$:



$\mathbb{R}$ A *right derived functor* of F is a pair $(\mathbb{R}F, \alpha)$ consisting of a functor $\mathbb{R}F : \mathscr{M}[\mathscr{W}^{-1}] \to \mathcal{D}$ and a natural transformation $\alpha : F \to \mathbb{R}F \circ p$, which furthermore is universal in the following sense. For every pair (G, β) of $G : \mathscr{M}[\mathscr{W}^{-1}] \to \mathcal{D}$ and $\beta : F \to G \circ p$ there is a unique natural transformation $\bar{\beta} : \mathbb{R}F \to G$ such that $\beta = (\bar{\beta} * p) \circ \alpha$:



ABS A left (right) derived functor $(\mathbb{L}F, \alpha)$ (resp. $(\mathbb{R}F, \alpha)$) of F is called *absolute* if for any functor $H : \mathcal{D} \to \mathcal{E}$ the pair $(H \circ \mathbb{L}F, H * \alpha)$ (resp. $(H \circ \mathbb{R}F, H * \alpha)$) is a left (right) derived functor of $H \circ F$.

Remark 4.10. It can be shown that the definition of localisation implies that the functor $p^* : Fun(\mathscr{M}[\mathscr{W}^{-1}], \mathcal{D}) \to Fun(\mathscr{M}, \mathcal{D})$ is fully faithful[15]. Thus it means that we could equivalently

[14] this was an example of complsing id_F and α.

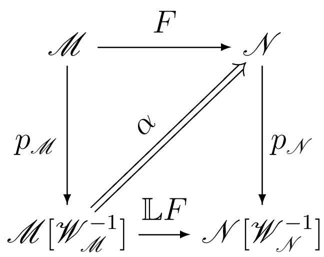
[15]Its image consists of morphisms of localisers $(\mathscr{M}, \mathscr{W}) \to (\mathcal{D}, Iso_{\mathcal{D}})$. In the weakened version one makes the replacement of *image* with *essential image*.

specify a left derived functor of F as a functor $\mathbb{L}F : \mathscr{M} \to \mathcal{D}$, which takes $\mathscr{W}$ to isomorphisms, and which is equipped with a natural transformation $\alpha : \mathbb{L}F \to F$ which has the universal property as above. This is however slightly less natural for *total* derived functors:

Definition 4.11. Let $(\mathscr{M}, \mathscr{W}_{\mathscr{M}})$ and $(\mathscr{N}, \mathscr{W}_{\mathscr{N}})$ be a pair of localizers $F : \mathscr{M} \to \mathscr{N}$ be a functor. An *total left derived functor* of F is a left derived functor of $p_{\mathscr{N}} \circ F$, where $p_{\mathscr{N}} : \mathscr{N} \to \mathscr{N}[\mathscr{W}_{\mathscr{N}}^{-1}]$ is the localization functor. Dually, an *total right derived functor* of F is a right derived functor of $p_{\mathscr{N}} \circ F$.

Remark 4.12. The notion of absolute total left (right) derived functor is defined in a straightforward manner.

We modify notation in this case so that an total left derived functor of F consists of a pair $(\mathbb{L}F, \alpha)$ where $\mathbb{L}F : \mathscr{M}[\mathscr{W}_{\mathscr{M}}^{-1}] \to \mathscr{N}[\mathscr{W}_{\mathscr{N}}^{-1}]$ and $\alpha : \mathbb{L}F \circ p_{\mathscr{M}} \to p_{\mathscr{N}} \circ F$. We could picture it as follows:



We shall often employ a short notation and write $\mathbb{L}F$ for an (total) left derived functor of F without mentioning α explicitly.

There are various reasons why derived functors are useful. The formal one is the following:

Theorem 4.13 (Abstract adjunction theorem). *Let $(\mathscr{M}, \mathscr{W}_{\mathscr{M}})$ and $(\mathscr{N}, \mathscr{W}_{\mathscr{N}})$ be a pair of localizers $F : \mathscr{M} \rightleftarrows \mathscr{N} : G$ be a pair of adjoint functors ($F \dashv G$). Then*

1. *If F admits an* absolute *total left derived functor $\mathbb{L}F$ And G admits an* absolute *total right derived functor $\mathbb{R}G$ then there is an adjunction $\mathbb{L}F \dashv \mathbb{R}G$ canonically induced from $F \dashv G$.*

2. *If F admits an absolute total left derived functor $\mathbb{L}F$ and there is a left adjoint H of $\mathbb{L}F$, then absolute total right derived functor $\mathbb{R}G$ of G exists and it can be chosen equal to H.*

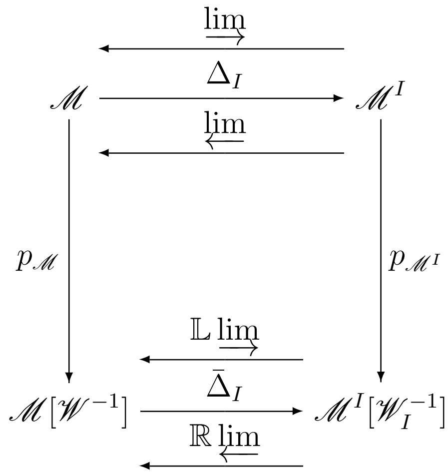
Proof. For the first statement, see [4] (the proof is a lengthy calculation; also, the explicit choices of unit and counit for the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ are given there). For the second statement, see [5]. However, the proof is really [2] a simple manipulation with Kan extensions and ends and will be included in future version. ☐

Thus we observe that (absolute) derived functors is a correct way to extend functors to localizations so that adjunctions are preserved. This fact lies in the basis of the next definition.

Let $(\mathscr{M}, \mathscr{W})$ be a localizer and $(\mathscr{M}^I, \mathscr{W}_I)$ be the localizer structure induced on $Fun(I, M)$ for a small category I as in Example 4.2. Then the functor Δ_I induces a functor $\bar{\Delta}_I : \mathscr{M}[\mathscr{W}^{-1}] \to \mathscr{M}^I[\mathscr{W}_I^{-1}]$ (indeed, $\bar{\Delta}_I$ is both absolute left and right derived functor of Δ_I.

Definition 4.14. A *homotopy colimit functor* is a left adjoint of $\bar{\Delta}_I$. Equivalently, it is a total left derived functor of $\varinjlim : \mathscr{M}^I \to \mathscr{M}$ (if the latter exists). A *homotopy limit functor* is a right adjoint of $\bar{\Delta}_I$. Equivalently, it is a total right derived functor of $\varprojlim : \mathscr{M}^I \to \mathscr{M}$ (if the latter exists).

We could picture this situation by the following diagram:



Example 4.15. The following example might be of some interest for a reader familiar with homological algebra. We shall reprove the existence of all functors in this example later in the course.

Take $\mathbf{DGMod}_R$ with $\mathscr{W}$ as in Example 4.3 (DG). Then for $I = \mathbf{B}G$, where G is a (finite) group we have $Fun(\mathbf{B}G, \mathbf{DGMod}_R) = \mathbf{DGMod}_{R[G]}$, that is, the category of complexes of left $R[G]$-modules. Here the ring $R[G]$ is the group R-algebra[16] of G.

The functor $\Delta_I : \mathbf{DGMod}_R \to \mathbf{DGMod}_{R[G]}$ sends a complex of R-modules $M^\bullet$ to the complex of $R[G]$-modules with trivial action of G. We observe from the Example 3.29 that the limit functor sends a complex of $R[G]$-modules $N^\bullet$ to the complex of invariants $(N^\bullet)^G$, and the colimit functor sends it to the complex of coinvariants $N^\bullet_G$.

An alternative way to express it is the following. There is a ring homomorphism $R[G] \to R$ which sends $\sum_{g\in G} a_g g$ to $\sum_{g\in G} a_g$. This homomorphism makes R in a $R[G]$ bimodule, that is, both left and right $R[G]$ module such that the actions are multiplicatively independent[17]. We then observe that, for a left $R[G]$-module N (or a complex of left $R[G]$-modules), we have $N_G = R \otimes_{R[G]} N$ and $N^G = Mod_{R[G]}(R, N)$.

In this case, we arrive to the fact that, for $N^\bullet \in \mathbf{DGMod}_{R[G]}$, we have (in classical homological algebra notation) $\mathbb{R}\varprojlim N^\bullet = \mathbb{R}Hom(R, N^\bullet)$, which is also known as $H^*(G, N^\bullet)$, the cohomology of group G with coefficients in $N^\bullet$, and $\mathbb{L}\varinjlim N^\bullet = R \otimes^{\mathbb{L}}_{R[G]} N^\bullet$, which is also known as $H_*(G, N^\bullet)$, the homology of group G with coefficients in $N^\bullet$.

With that kind of formal introduction we are left with the following list of problems:

1. We were somewhat able to prove that localization exists, but what about derived functors?

[16] As a set, it consists of all R-linear combinations of elements from G. The multiplication is induced from that of G.

[17] Another way of saying this is that R has the structure of left $R[G] \otimes_R R[G]^{op}$-module, where, for a ring A, A^{op} denotes the ring with opposite multiplication.

Under what consequences does a functor between categories with weak equivalences admit a left or right derived functor?

2. If F and G a couple of functors between localisers, and if $\mathbb{L}F$ and $\mathbb{L}G$ are their total left derived functors, the derived functor $\mathbb{L}(G \circ F)$ might not exist, but even if it does, the canonical natural transformation $\mathbb{L}G \circ \mathbb{L}F \to \mathbb{L}(G \circ F)$ might not be an isomorphism. Under what circumstances does it happen?

3. When do homotopy (co)limits exist for a localizer $(\mathscr{M}, \mathscr{W})$?

4. How to compute all this stuff?

It is now time to propose an answer to these questions.

5 Model Categories

5.1 Definition and examples

Definition 5.1. A *closed model structure* on a category $\mathcal{M}$ consists of three (not necessarily small) sets of morphisms $\mathcal{W}$, Fib and Cof (that is, all three are subsets of $\operatorname{Mor} \mathcal{M}$), called respectively *weak equivalences, fibrations* and *cofibrations*, such that the folliwing axioms are satisfied:

CM1 (3-for-2) If $f : X \to Y$, $g : Y \to Z$ are morphisms in $\operatorname{Mor} \mathcal{M}$ and it is true that any 2 elements of the set $\{f, g, g \circ f\}$ are in $\mathcal{W}$, then $\{f, g, g \circ f\} \subset \mathcal{W}$.

CM2 (Retracts) Given a commutative diagram

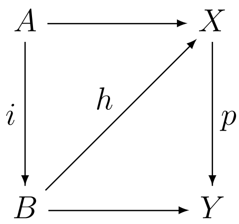
$$\begin{array}{ccccc} A & \xrightarrow{i_1} & X & \xrightarrow{r_1} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{i_2} & Y & \xrightarrow{r_2} & B \end{array}$$

in which $r_1 \circ i_1 = id_A$ and $r_2 \circ i_2 = id_B$ (in such case f is called a *retract* of g), if $g \in \mathcal{W}$ (respectively $g \in Fib$, $g \in Cof$) then $f \in \mathcal{W}$ (respectively $f \in Fib$, $f \in Cof$).

CM3 (Lifting) Given a commutative diagram

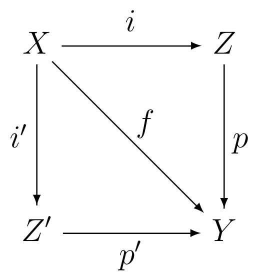
$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

Whenever $i \in Cof$, $p \in Fib \cap \mathcal{W}$ (or $i \in Cof \cap \mathcal{W}$, $p \in Fib$), there exists a morphism $h : B \to X$ such that the resulting diagram



commutes.

CM4 (Factorization) Any morphism $f : X \to Y$ can be factored in two ways,



such that $i \in Cof \cap \mathcal{W}$, $p \in Fib$ and $i' \in Cof$, $p' \in Fib \cap \mathcal{W}$.

Morphisms from $Cof \cap \mathcal{W}$ are called *trivial*, or *acyclic*, cofibrations. Morphisms from $Fib \cap \mathcal{W}$ are called *trivial*, or *acyclic*, fibrations.

Definition 5.2. A *model category* is a category $\mathcal{M}$ with a model structure $\mathcal{W}, Fib, Cof$ such that

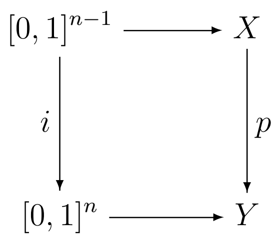
CM0 $\mathcal{M}$ admits small limits and colimits.

In a model category $\mathcal{M}$ one has an initial object $\emptyset$ and a final object $*$. An object X in $\mathcal{M}$ is called *cofibrant* if the morphism $\emptyset \to X$ is a cofibration and is called *fibrant* if the morphism $X \to *$ is a fibration.

Example 5.3.

TRIV Any category $\mathcal{C}$ can be given *trivial* model structure with $\mathcal{W} = Iso(\mathcal{C})$ and $Fib = Cof = \operatorname{Mor} \mathcal{C}$.

TOP A *Serre fibration* is a map of topological spaces $p : X \to Y$ such that for every $n \geq 0$ and every commutative diagram



there exists a map $h : [0,1]^n \to X$ such that the diagram obtained from the square above by adding h commutes. Here $[0,1]$ is the standard interval in $\mathbb{R}$ and $i : [0,1]^{n-1} \to [0,1]^n$

includes $[0,1]^{n-1}$ as one of the faces of $[0,1]^n$. Coverings and fibre bundles are examples of Serre fibrations, and another example of such a map is the morphism $p : Path(X) \to X \times X$. Here $Path(X)$ is the space of triples (x, y, l), where $x, y \in X$ and $l : [0,1] \to X$ is a path[18] in X. p then is the obvious projection. We might restrict ourselves to $x \times X \subset X \times X$ and obtain, by taking the limit of the diagaram

$$\begin{array}{ccc} & & Path(X) \\ & & \downarrow p \\ x \times X & \hookrightarrow & X \times X \end{array}$$

the map $e : SX \to X$. Here SX is the space of all maps $l : [0,1] \to X$ such that $l(0) = x$ and $e(l) = l(1)$. This is also a Serre fibration.

There is a model structure on **Top** in which Fib is the set of all Serre fibrations and $\mathcal{W}$ is the set of all weak homotopy equivalences. Cof in this model structure are not easy to describe. Informally, they are retracts of inclusions $X \hookrightarrow Y$ such that Y is obtained from X by attaching (perhaps infinite number of) cells (that is, disks D^n of any dimension). Trivial cofibrations, however, are known to be inclusions $X \hookrightarrow Y$ which are strict deformation retracts.

Every object is fibrant in this model structure. Cofibrant objects could be called 'generalized cell complexes'.

SSET The category of simplicial sets admits a very sophisticated model structure. First, define simplicial subsets $\Lambda^n_k \subset \Delta^n$ (here $k \in \{0, ..., n\}$) as follows. Take all non-degenerate simplices of Δ^n, and remove from this list $id_{[n]} : [n] \to [n]$ and $\partial_k : [n-1] \to [n]$ (the latter is the unique injective order-preserving map whose image does not contain k). Then consider the minimal simplicial subset of Δ^n containing the remaining non-degenerate simplices. Alternatively, take the colimit in $SSet$ of the category of elements of Δ^n modulo $id_{[n]}$ and ∂_k. Either way we finish up with a simplicial set Λ^n_k with an (injective) map $\Lambda^n_k \to \Delta^n$. From the construction, we observe that the geometric realization of Λ^n_k is the boundary of n-simplex with kth face removed.

A *Kan fibration* is a map of simplicial sets $p : X \to Y$ such that for any $n \in \mathbb{N}$ and $k \in \{0, ..., n\}$ and any commutative diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y \end{array}$$

(where i is the canonical inclusion) there exists a morphism $h : \Delta^n \to X$ such that the diagram

[18] We shall not dwell in details on the topology of $Path(X)$, assuming this problem is treated in topology courses.

obtained from the square above by adding h commutes. An example of Kan fibration is the map $Sing(T) \to *$, where T is a topoligical space and $*$ is the final simplicial set. Indeed, finding a lift[19] $\Delta^n \to Sing(T)$ in the diagram

$$\begin{CD} \Lambda^n_k @>>> Sing(T) \\ @V{i}VV @VVV \\ \Delta^n @>>> * \end{CD}$$

which is, by the way, the same as finding a lift $\Delta^n \to Sing(T)$ in

$$\begin{CD} \Lambda^n_k @>>> Sing(T) \\ @V{i}VV \\ \Delta^n \end{CD}$$

is equivalent, by adjointness $|-| \dashv Sing$, to finding a lift $|\Delta^n| \to T$ in

$$\begin{CD} |\Lambda^n_k| @>>> T \\ @V{i}VV \\ |\Delta^n| \end{CD}$$

But $|\Lambda^n_k| \cong D^{n-1}$, $|\Delta^n| \cong D^n$ and any topological space is obviously Serre-fibrant. We shall discover other fibrant simplicial sets later in the course.

To obtain a model structure, one defines $\mathcal{W}$ as in Example 4.3 and also Cof to be all monomorphisms and Fib to be all Kan fibrations. All objects are cofibrant in this model structure. The fibrant objects are called *Kan complexes*.

PROJ Let $\mathbf{DGMod}_A^{\leq 0}$ be the category of complexes of A-modules such that for any $N^\bullet \in \mathbf{DGMod}_A^{\leq 0}$ we have $N^i = 0$ for $i > 0$. There is a model structure on this category in which

- $\mathcal{W}$ are quasiisomorphisms,
- Cof are maps $f : N^\bullet \to M^\bullet$ such that $f^i : N^i \to M^i$ is an injection whose cokernel is a projective module,
- Fib are maps $f : N^\bullet \to M^\bullet$ such that for $i < 0$ $f^i : N^i \to M^i$ is a surjection.

This model structure is called the *projective model structure* on $\mathbf{DGMod}_A^{\leq 0}$. All objects are fibrant and cofibrant objects are precisely the complexes of projective A-modules bounded from the right.

[19]That is, a diagonal map so that the resulting diagram commutes.

INJ Let $\mathbf{DGMod}_A^{\geq 0}$ be the category of complexes of A-modules such that for any $N^\bullet \in \mathbf{DGMod}_A^{\geq 0}$ we have $N^i = 0$ for $i < 0$. There is a model structure on this category in which

- $\mathcal{W}$ are quasiisomorphisms,
- Cof are maps $f : N^\bullet \to M^\bullet$ such that for $i > 0$ $f^i : N^i \to M^i$ is an injection,
- Fib are maps $f : N^\bullet \to M^\bullet$ such that for $f^i : N^i \to M^i$ is a surjection whose kernel is an injective module.

This model structure is called the *injective model structure* on $\mathbf{DGMod}_A^{\leq 0}$. All objects are cofibrant and fibrant objects are precisely the complexes of injective A-modules bounded from the left.

DG The category $\mathbf{DGMod}_A$ has two model structures, which are in a sense inherited from the previous two examples. In the first one

- $\mathcal{W}$ are quasiisomorphisms,
- Cof are maps $f : N^\bullet \to M^\bullet$ such that $f^i : N^i \to M^i$ is an injection and the cokernel of f is a DG-projective complex.
- Fib are maps $f : N^\bullet \to M^\bullet$ such that for $f^i : N^i \to M^i$ is a surjection.

We shall again call it the *projective model structure*. In the second one

- $\mathcal{W}$ are quasiisomorphisms,
- Cof are maps $f : N^\bullet \to M^\bullet$ such that for $f^i : N^i \to M^i$ is an injection,
- Fib are maps $f : N^\bullet \to M^\bullet$ such that for $f^i : N^i \to M^i$ is a surjection and the kernel of f is a DG-injective complex.

This one is again to be called the *injective model structure*. We shall not explain here what 'DG-projective' and 'DG-injective' means as we won't need such explicit descriptions of (co)fibrations.

It is important to note here that in both model structures the choice of $\mathcal{W}$ is the same, so, if we consider $\mathbf{DGMod}_A[\mathcal{W}^{-1}]$ for both model structures, we shall get equivalent categories. However, different model structures are useful for different purposes. For instance the derived tensor product functor will be seen to exist in projective model structure, and the derived Hom-functor works well with injective model structure. In this sense, choosing model structure is like choosing coordinates for computation. This analogy becomes even more drastic when one considers homotopy (co)limits.

CAT Lastly, for the category of small categories **Cat** we have the following *canonical* model structure in which

- $\mathcal{W}$ are equivalences of categories,

– *Fib* are *isofibrations*, that is, functors $F : \mathcal{C} \to \mathcal{D}$ such that for any object X in $\mathcal{C}$ and an isomorphism $u : F(X) \to Y$ in $\mathcal{D}$ there exists an isomorphism $\tilde{u} : X \to \widetilde{Y}$ in $\mathcal{C}$ with $F(\tilde{u}) = u$,

– *Cof* are functors injective on the sets of objects.

Any object is fibrant and cofibrant in this model structure.

Let us elaborate a bit more on the final example.

Proposition 5.4 ([6]). *The category* **Cat** *with* $\mathcal{W}, Fib, Cof$ *as above is a model category.*

Proof. Let us verify all the steps.

CM0 The category **Cat** is complete (this is easy to see) and cocomplete (this is not so easy to see; coequalizers are usually not calculated explicitly).

CM1 This property is evident for functors which are equivalences of categories.

CM2 Let

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{I_1} & \mathcal{M} & \xrightarrow{R_1} & \mathcal{C} \\ {\scriptstyle F}\downarrow & & {\scriptstyle G}\downarrow & & {\scriptstyle F}\downarrow \\ \mathcal{D} & \xrightarrow{I_2} & \mathcal{N} & \xrightarrow{R_2} & \mathcal{D} \end{array}$$

be a retract diagram of small categories and functors.

If $G \in \mathcal{W}$, choose a quasi-inverse $G' : \mathcal{N} \to \mathcal{M}$ of G and consider the functor $R_1 \circ G' \circ I_2 : \mathcal{D} \to \mathcal{C}$. This will be a quasi-inverse of F.

For *Fib* this will follow from some general arguments to be outlined below. For *Cof*, the fact that in the diagram above $F \in Cof$ whenever $G \in Cof$ is equivalent to the fact that in the diagram

$$\begin{array}{ccccc} \operatorname{Ob}\mathcal{C} & \xrightarrow{I_1} & \operatorname{Ob}\mathcal{M} & \xrightarrow{R_1} & \operatorname{Ob}\mathcal{C} \\ {\scriptstyle F}\downarrow & & {\scriptstyle G}\downarrow & & {\scriptstyle F}\downarrow \\ \operatorname{Ob}\mathcal{D} & \xrightarrow{I_2} & \operatorname{Ob}\mathcal{N} & \xrightarrow{R_2} & \operatorname{Ob}\mathcal{D} \end{array}$$

'$G : \operatorname{Ob}\mathcal{M} \to \operatorname{Ob}\mathcal{N}$ is injective' implies '$F : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ is injective'. This can be checked by a direct calculation.

CM3 Consider a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{U} & \mathcal{M} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{D} & \xrightarrow{V} & \mathcal{N} \end{array}$$

of small categories and functors. We check that there exists a lift $H : \mathcal{D} \to \mathcal{M}$ whenever $F \in Cof \cap \mathcal{W}, G \in Fib$ or $F \in Cof, G \in Fib \cap \mathcal{W}$.

WCOF When F is a cofibration and a weak equivalence, we see that, in this case, $\mathcal{C}$ may be identified with a full subcategory $F(\mathcal{C})$ of $\mathcal{D}$ having the property that $\operatorname{Ob} F(\mathcal{C}) = F(\operatorname{Ob} \mathcal{C})$ and that every object of $\mathcal{D}$ is isomorphic to an object of $F(\mathcal{C})$. We may use this to see that there exists a functor $F' : \mathcal{D} \to \mathcal{C}$ with the list of properties

1. $F' \circ F = id_{\mathcal{C}}$.
2. There is a natural isomorphism $\alpha : F \circ F' \xrightarrow{\sim} id_{\mathcal{D}}$ such that for every $c \in \mathcal{C}$ we have $\alpha_{Fc} = id_{Fc}$.

(we leave it to the reader to elaborate on the existense of such a functor F')

We now construct a lift explicitly. For every $d \in \mathcal{D}$ there is an isomorphism $V\alpha_d : V \circ F \circ F'd \xrightarrow{\sim} Vd$ in $\mathcal{N}$. But $V \circ F \circ F' = G \circ U \circ F'$ so α_d gives an isomorphism $G(U \circ F'd) \xrightarrow{\sim} Vd$. But $G \in Fib$ so, by the property of fibrations, there is a lift

$$\beta_d : U \circ F'd \xrightarrow{\sim} Hd.$$

Let us chose a lift β_d for every $d \in \mathcal{D}$.

If $f : d \to d'$ is a morphism in $\mathcal{D}$, then we define Hf to be the composite

$$Hd \xrightarrow{\beta_d^{-1}} U \circ F'd \xrightarrow{U \circ F'f} U \circ F'd' \xrightarrow{\beta_{d'}} Hd'.$$

It can be checked that this amount of data defines a functor $H : \mathcal{D} \to \mathcal{M}$ such that everything commutes.

WFIB The fact that G is a fibration and a weak equivalence means that G is fully faithful and surjective on objects. For every $d \in \mathcal{D}$, consider $Vd \in \mathcal{N}$ and its lift $Hd \in \mathcal{M}$ (this exists as G is surjective on objects). Next, as G is fully faithful,

$$\mathcal{M}(Hd, Hd') \cong \mathcal{N}(Vd, Vd') \leftarrow \mathcal{D}(d, d')$$

So the fully-faithfulness gives us a map $\mathcal{D}(d, d') \to \mathcal{M}(Hd, Hd')$ which in total gives us a functor $H : \mathcal{D} \to \mathcal{M}$.

CM4 We must factor a functor $F : \mathcal{C} \to \mathcal{D}$ in two ways,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;I\;} & \mathcal{E} \\ {\scriptstyle I'}\downarrow & \searrow^{F} & \downarrow{\scriptstyle P} \\ \mathcal{E}' & \xrightarrow[\;P'\;]{} & \mathcal{D} \end{array}$$

such that $I \in Cof$, $P \in Fib \cap \mathcal{W}$ and $I' \in Cof \cap \mathcal{W}$, $P' \in Fib$.

E Let us first factor F as $P \circ I$. To do this, construct the following category $\mathcal{E}$. Set

$$\mathrm{Ob}\,\mathcal{E} = \{(c, \alpha, d) | c \in \mathcal{C},\, d \in \mathcal{D},\, \alpha : Fc \xrightarrow{\sim} d\}.$$

Here α is an isomorphism between Fc and d. For two objects, (c, α, d) and (c', α', d'), we set

$$\mathcal{E}((c, \alpha, d), (c', \alpha', d')) = \mathcal{C}(c, c').$$

There is a functor

$$I : \mathcal{C} \to \mathcal{E},\ c \mapsto (c, \mathsf{id}_{Fc}, Fc)$$

which is seen to be a cofibration and a weak equivalence, and also a functor

$$P : \mathcal{E} \to \mathcal{D},\ (c, \alpha, d) \mapsto d$$

which is seen to be a fibration.

E' Define $\mathcal{E}'$ to be the category with

$$\mathrm{Ob}\,\mathcal{E}' = \mathrm{Ob}\,\mathcal{C} \sqcup \mathrm{Ob}\,\mathcal{D}$$

and for $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$ set

$$\mathcal{E}'(c, c') = \mathcal{C}(c, c');\ \mathcal{E}'(c, d') = \mathcal{D}(Fc, d'),$$

$$\mathcal{E}'(d, c') = \mathcal{D}(d, Fc'),\ \mathcal{E}'(d, d') = \mathcal{D}(d, d').$$

There is a functor

$$I' : \mathcal{C} \to \mathcal{E}',\ c \in \mathrm{Ob}\,\mathcal{C} \mapsto c \in \mathrm{Ob}\,\mathcal{E}',$$

which is evidently a cofibration, and a functor

$$P' : \mathcal{E}' \to \mathcal{D}$$

which maps $c \in \mathrm{Ob}\,\mathcal{C} \subset \mathrm{Ob}\,\mathcal{E}'$ to $Fc \in \mathcal{D}$ and $d \in \mathrm{Ob}\,\mathcal{D} \subset \mathrm{Ob}\,\mathcal{E}'$ to $d \in \mathcal{D}$. This functor is evidently a fibration and a weak equivalence. □

This example is a rare opprotunity to prove all the axioms by presenting explicit lifts and factorizations.

5.2 Lifting and retract argument

We turn to the original meaning of the word 'closed'.

Let $\mathcal{C}$ be a category with $I \subset \operatorname{Mor} \mathcal{C}$.

Definition 5.5.

RLP The set of morphisms with *right lifting property* (RLP) with respect to I is denoted by $r(I)$ and consists of all morphisms $p : X \to Y$ in $\operatorname{Mor} \mathcal{C}$ such that for any $i : A \to B \in I$ and any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

there exists a *lift*, that is, a map $h : B \to X$ such that the resulting diagram

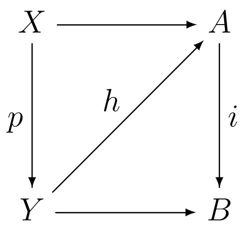
$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

commutes.

LLP The set of morphisms with *left lifting property* (LLP) with respect to I is denoted by $l(I)$ and consists of all morphisms $p : X \to Y$ in $\operatorname{Mor} \mathcal{C}$ such that for any $i : A \to B \in I$ and any commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \longrightarrow & B \end{array}$$

there exists a *lift*, that is, a map $h : Y \to A$ such that the resulting diagram



commutes.

Example 5.6. Most fibrations we have seen in Example 5.3 arise as morphisms with right lifting property with respect to some set of maps.

TOP In the category **Top**, Serre fibrations are elements of the set $r(I)$, where I is the set of all face inclusions $[0,1]^{n-1} \to [0,1]^n$ for $n \geq 0$.

SSET In the category **SSet**, Kan fibrations are elements of the set $r(I)$, where I is the set of all horn inclusions $\Lambda^n_k \to \Delta^n$ for $n \geq 0$ and $0 \leq k \leq n$.

CAT In the category **Cat**, isofibrations are elements of the set $r(I)$, where I is the set consisting of one element, the morphism

$$[0] \xrightarrow{0} [1]_{iso}$$

(here $[1]_{iso}$ is the 'one-isomorphism' category as in the proof of Proposition 4.6).

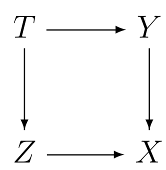
Proposition 5.7. *For a set of morphisms $I \subset \operatorname{Mor} \mathcal{C}$,*

1. *The set $r(I)$ is stable under composition, pullbacks and retracts,*

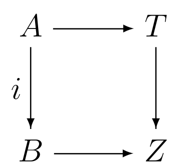
2. *The set $l(I)$ is stable under composition, pushouts and retracts.*

Proof. We shall prove the case of $r(I)$ (for $l(I)$, the proof is dual).

- Let



Be a pullback diagram, that is, $T \cong \varprojlim(Z \to X \leftarrow Y)$. For $i : A \to B$ in I, consider a (commutative) diagram



In total, this gives us

$$\begin{array}{ccccc} A & \longrightarrow & T & \longrightarrow & Y \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & Z & \longrightarrow & X \end{array}$$

There is a lift

$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow \\ B & \longrightarrow & X \end{array}$$

As $B \to X$ is a composite morphism $B \to Z \to X$, there is a morphism $B \to T$ given by the universal property of T. This is seen to be a desired lift.

- Let

$$\begin{array}{ccccc} Z & \xrightarrow{i_1} & X & \xrightarrow{r_1} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ T & \xrightarrow{i_2} & Y & \xrightarrow{r_2} & T \end{array}$$

be a retract diagram so that $g \in r(I)$. A lifting problem for f, that is, finding a lift in a diagram

$$\begin{array}{ccc} A & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & T \end{array}$$

can be done by first constructing the diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z & \xrightarrow{i_1} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & T & \xrightarrow{i_2} & Y \end{array}$$

and finding a lift $h : B \to X$. Then the morphism $r_1 \circ h : B \to Z$ can be checked to be the required lift.

- The proof for composition and the (dual) proof for $l(I)$ is left to the reader. □

One final property is the *retract argument*:

Proposition 5.8. *Let $I \subset \operatorname{Mor} \mathcal{C}$. If an element $f : X \to Y$ of the set I can be factored as $f = p \circ i$ where $p \in r(I)$ (respectively, $i \in l(I)$) then f is a retract of i (respecitvely f is a retract of p).*

Proof. Let us prove the first part. Since we can factor $f = p \circ i$, there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{id_Y} & Y \end{array}$$

Since $p \in r(I)$ and $f \in I$ then there is a lift $h : Y \to Z$ such that $p \circ h = id_Y$ and $h \circ f = i = i \circ id_X$. This lift allows us to draw a diagram

$$\begin{array}{ccccc} X & \xrightarrow{id_X} & X & \xrightarrow{id_X} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{h} & Z & \xrightarrow{p} & Y \end{array}$$

which shows that f is a retract of i. □

These two propositions allow us to deduce the following:

Theorem 5.9. *In a model category $\mathcal{M}$ with fibrations Fib, cofibrations Cof and weak equivalences $\mathcal{W}$,*

1. $Cof = l(Fib \cap \mathcal{W})$, $Cof \cap \mathcal{W} = l(Fib)$.
2. $Fib = l(Cof \cap \mathcal{W})$, $Fib \cap \mathcal{W} = l(Cof)$.
3. *Cof and $Cof \cap \mathcal{W}$ are stable under composition, pushouts and retracts*
4. *Fib and $Fib \cap \mathcal{W}$ are stable under composition, pullbacks and retracts.*

Proof. The inclusion $Cof \subset l(Fib \cap \mathcal{W})$ is evident from the definition, the inverse inclusion is obtained by **CM4**, **CM3** and Proposition 5.8. This is the way to prove 1. and 2.; the proof of 3. and 4. is done by using Proposition 6.1.2. □

In fact, we see that the axiom **CM2** is a bit overdetermined. We do not need to require the stability under retracts for Fib and Cof as it automatically follows from the theorem we just proved.

5.3 Homotopy in a model category

Definition 5.10. Let $\mathcal{M}$ be a model category. Its *homotopy category* $\operatorname{Ho}\mathcal{M}$ is the localisation of $\mathcal{M}$ with respect to weak equivalences $\mathcal{W}$.

The purpose of this section is to provide a description of $\operatorname{Ho}\mathcal{M}$ which exists without moving to a higher universe and is much more tractable. This is done via the notion of *homotopy* in a model category.

Fix a model category $\mathcal{M}$ with fibrations Fib, cofibrations Cof and weak equivalences $\mathcal{W}$.

Definition 5.11.

- For $X \in \mathcal{M}$, a *cylinder object* of X is the factorization of the codiagonal morphism,

$$\begin{array}{ccc} X \coprod X & \xrightarrow{\nabla} & X \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle p} & \\ C(X) & & \end{array}$$

  such that p is a weak equivalence. We shall denote this cylinder object by $C(X)$ omitting p and i.

  A cylinder object is called *good* if $i \in Cof$. A cylinder object is called *very good* if in addition $p \in Fib\mathcal{W}$.

- For $Y \in \mathcal{M}$, a *path object* of Y is the factorization of the diagonal morphism,

$$\begin{array}{ccc} Y & \xrightarrow{\Delta} & Y \times Y \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle p} & \\ P(Y) & & \end{array}$$

  such that i is a weak equivalence. We shall denote this path object by $P(Y)$ omitting p and i.

  A path object is called *good* if $p \in Fib$. A path object is called *very good* if in addition $i \in Cof \cap \mathcal{W}$.

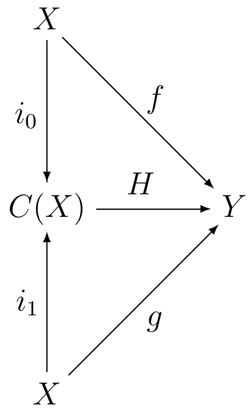
Remark 5.12. By **CM4**, in a model category every object X has both a very good cylinder and a very good path objects. We also introduce some notation.

Let X, i and p be as in the definition of $C(X)$ above. There are two morphisms $in_0, in_1 : X \to X \coprod X$ given by the structure of coproduct. We denote by i_0 and i_1 the compositions $i \circ in_0$ and $i \circ in_1$. This allows us to write $i = i_0 \sqcup i_1$.

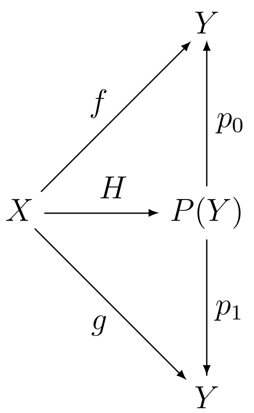
Dually, let Y, p and i be as in the definition of $P(Y)$ above there are two morphisms $pr_0, pr_1 : Y \times Y \to Y$ given by the structure of product. We denote by p_0 and p_1 the compositions $p \circ pr_0$ and $p \circ pr_1$. This allows us to write $p = (p_0, p_1)$.

Definition 5.13. Let $f, g : X \to Y$ be two morphisms in $\mathcal{M}$.

LH f and g are *left homotopic* (in this case we write $f \overset{l}{\sim} g$) if there is a cylinder object $C(X)$ of X and a morphism $H : C(X) \to Y$ such that $H \circ i_0 = f$, $H \circ i_1 = g$ (in this case, H is called a *left homotopy* from f to g):



RH f and g are *right homotopic* (in this case we write $f \overset{r}{\sim} g$) if there is a cylinder object $P(Y)$ of Y and a morphism $H : X \to P(Y)$ such that $p_) \circ H = f$, $p_1 \circ H = g$ (in this case, H is called a *right homotopy* from f to g):



H f and g are *homotopic* (in this case we write $f \sim g$ if they are both left and right homotopic.

HEQ f is a *homotopy equivalence* if there exists a morphism $f' : Y \to X$ such that $f \circ f' \sim id_Y$ and $f' \circ f \sim id_X$.

Definition 5.14. For $X, Y \in \mathcal{M}$, denote by

- $\pi^l(X, Y)$ (the set of all *left homotopy classes*) to be the quotient of $\mathcal{M}(X, Y)$ by the equivalence relation which is minimal among all equivalence relations on $\mathcal{M}(X, Y)$ containing the 'elementary relations' of the form $f \overset{l}{\sim} g$.

- $\pi^r(X, Y)$ (the set of all *right homotopy classes*) to be the quotient of $\mathcal{M}(X, Y)$ by the equivalence relation which is minimal among all equivalence relations on $\mathcal{M}(X, Y)$ containing the 'elementary relations' of the form $f \overset{r}{\sim} g$.

Remark 5.15. Aside from the notion of (very) good cylinder and path objects, all the definitions outlined could be made only with finite (co)products and $\mathcal{W}$ without any mention of Fib and Cof. However, the abstract notion of homotopy gives some meaningful results only for a model category.

We now outline the main properties of left and right homotopy which allow us to present a meaningful description of $\operatorname{Ho}\mathcal{M}$. The facts are multiple and most proofs are diagram chase. We present them in a unified proposition for easier reference, and write all the results for the case of a left homotopy (the case of the right homotopy follows by reversing arrows and replacing 'fibration' with 'cofibration' and vice versa).

Proposition 5.16. *Let*[20] $f, g \in \mathcal{M}(X, Y)$.

1. $f \overset{l}{\sim} g$ *iff there exists a good cylinder object* $C(X)$ *and a left homotopy* $H : C(X) \to Y$. *If, in addition,* Y *is fibrant, then* $C(X)$ *above can be chosen to be very good.*
2. *If* X *is cofibrant and* $C(X)$ *is a good cylinder object of* X *then* $i_0, i_1 : X \to C(X)$ *are elements of* $Cof \cap \mathcal{W}$.
3. *If* X *is cofibrant then* $\overset{l}{\sim}$ *is an equivalence relation on* $\mathcal{M}(X, Y)$.
4. *If* X *is cofibrant then* $f \overset{l}{\sim} g$ *implies* $f \overset{r}{\sim} g$.
5. *If* $p : E \to B$ *is a fibration and a weak equivalence, and* $A \in \mathcal{M}$ *is cofibrant, then*
$$p_* : \pi^l(A, E) \to \pi^l(A, B),\ [f] \mapsto [p \circ f]$$
is an isomorphism.
6. *If* Y *is fibrant then for any* $Z \in \mathcal{M}$ *the composition*
$$\mathcal{M}(Z, X) \times \mathcal{M}(X, Y) \to \mathcal{M}(Z, Y)$$
induces a composition on the left homotopy classes:
$$\pi^l(Z, X) \times \pi^l(X, Y) \to \pi^l(Z, Y).$$
7. *(Whitehead theorem) If* X *and* Y *are both fibrant and cofibrant then* $f \in \mathcal{W}$ *iff* f *is a homotopy equivalence.*

Proof. Most proofs are quite elementary.

1. Let $C(X)$ be a cylinder object and $H : C(X) \to Y$ a left homotopy from f to g. Factor $i : X \coprod X \to C(X)$ as
$$X \coprod X \xrightarrow{i'} C'(X) \xrightarrow{p'} C(X)$$
so that $i' \in Cof$ (and $p' \in Fib \cap \mathcal{W}$). We then see that $C'(X)$ is a good cylinder object for X and $H \circ p' : C'(X) \to Y$ is a left homotopy from f to g.

[20] In this proposition we use the notation of Definition 5.11.

From now on, let $C(X)$ be a good cylinder object (so that i is a cofibration). If Y is fibrant, factor $p : C(X) \to X$ as

$$C(X) \xrightarrow{i''} C''(X) \xrightarrow{p''} X$$

so that $i'' \in Cof$ and $p'' \in Fib \cap \mathcal{W}$. This implies that $C''(X)$ is a very good cylinder object for X. Since $p = p'' \circ i''$, $p \in \mathcal{W}$ and $p'' \in \mathcal{W}$, **CM1** implies that $i'' \in Cof \cap \mathcal{W}$. We then get a diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{H} & Y \\ {\scriptstyle i''}\big\downarrow & & \\ C''(X) & & \end{array}$$

and, since Y is fibrant and i'' is a cofibration and a weak equivalence, we find a lift $H'' : C''(X) \to Y$. It is easily checked that this morphism is a left homotopy from f to g.

2. Consider the pushout diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle in_0} \\ X & \xrightarrow[in_1]{} & X \coprod X \end{array}$$

As $\emptyset \to X$ is a cofibration and Cof is stable under pushout, in_0 and in_1 are cofibrations. As, for a good cylinder object, $i_0 = i \circ in_0$ and in_1 are cofibrations. In addition, $p \circ in_0 = p \circ in_1 = id_X$ and[21] $p, id_X \in \mathcal{W}$ so, by **CM1**, i_0 and i_1 are cofibrations and weak equivalences.

3. The relation $\overset{l}{\sim}$ is seen to be reflexive and symmetric. To check the transitivity, let $f, g, h : X \to Y$. Chose two good cylinder objects

$$X \coprod X \overset{i_0 \sqcup i_1}{\longrightarrow} C(X) \xrightarrow{p} X, \quad X \coprod X \overset{i'_0 \sqcup i'_1}{\longrightarrow} C'(X) \xrightarrow{p'} X$$

and two left homotopies: $H : C(X) \to Y$ from f to g and $H' : C'(X) \to Y$ from g to h. Then consider the diagram

$$\begin{array}{ccccc} & & & & X \\ & & & & \big\downarrow{\scriptstyle i_0} \\ & & X & \xrightarrow{i_1} & C(X) \\ & & {\scriptstyle i'_0}\big\downarrow & & \\ X & \xrightarrow{i'_1} & C'(X) & & \end{array}$$

[21]It is easy to see that for every $T \in \mathcal{M}$ the morphism id_T is in $Fib \cap Cof \cap \mathcal{W}$ as it has both right and left lifting property with respect to any subset of $\mathrm{Mor}\, \mathcal{M}$.

and take its colimit. We then get a diagram

$$\begin{array}{ccccc} & & & & X \\ & & & & \big\downarrow{\scriptstyle i_0} \\ & & X & \xrightarrow{i_1} & C(X) \\ & & \big\downarrow{\scriptstyle i_0'} & & \big\downarrow{\scriptstyle a} \\ X & \xrightarrow{i_1'} & C'(X) & \xrightarrow{b} & C''(X) \end{array}$$

We want to show that $C''(X)$ is a cylinder object for X so that there is a left homotopy $H'' : C''(X) \to Y$ from $f \to h$.

There are inclusion morphisms $a \circ i_0$ and $b \circ i_1'$ from $X \to C''(X)$. This gives us a morphism $X \coprod X \to C''(X)$. Moreover, one can see that the universal property of a colimit gives us a map $p'' : C''(X) \to X$ such that $p'' \circ a = p$ and $p'' \circ b = p'$.

But a is a pushout of i_1 which is an element of $Cof \cap \mathcal{W}$ by (2) of this proposition. It implies that $a \in Cof \cap \mathcal{W}$. As $p \in \mathcal{W}$, by **CM1** we have $p'' \in \mathcal{W}$.

Lastly, the universal property of colimit implies that there is a morphism $H'' : C''(X) \to Y$ such that $H'' \circ a = H$, $H'' \circ b = H'$. This is checked to be the desired left homotopy from f to h.

4. Let

$$X \coprod X \overset{i_0 \sqcup i_1}{\longrightarrow} C(X) \overset{p}{\longrightarrow} X$$

be a good cylinder object and $H : C(X) \to Y$ be a left homotopy from f to g. Let

$$Y \overset{i}{\longrightarrow} P(Y) \overset{(p_0, p_1)}{\longrightarrow} Y \times Y$$

be a good path object for Y. Consider a diagram

$$\begin{array}{ccc} X & \xrightarrow{i \circ f} & P(Y) \\ \big\downarrow{\scriptstyle i_0} & & \big\downarrow{\scriptstyle (p_0, p_1)} \\ C(X) & \xrightarrow{(f \circ p, H)} & Y \times Y \end{array}$$

As, by (2) of this Proposition, $i_0 \in Cof \cap \mathcal{W}$ and $(p_0, p_1) \in Fib$, there is a lift $G : C(X) \to P(Y)$. Then $G \circ i_1$ is the desired right homotopy from f to g.

5. For any $f : A \to B$ we have a diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

As A is cofibrant there is a lift $g : A \to E$, $p \circ g = f$. This proves that p_* is surjective. If $g, g' : A \to E$ are two morphisms such that $p \circ g \overset{l}{\sim} p \circ g'$, then choose a good cylinder object $C(A)$ and left homotopy $H : C(A) \to B$. We then have a diagram

$$\begin{array}{ccc} A\coprod A & \xrightarrow{g \sqcup g'} & E \\ \downarrow & & \downarrow p \\ C(A) & \xrightarrow{H} & B \end{array}$$

The left map is a cofibration so we can find a lift $G : C(A) \to E$. This is a left homotopy from g to g'.

6. We need to check that the composition map

$$\mathcal{M}(Z, X) \times \mathcal{M}(X, Y) \to \mathcal{M}(Z, Y)$$

respects the elementary relation $\overset{l}{\sim}$ between morphisms. It is sufficient to see that

- If $f, g : X \to Y$ and $k : Z \to X$, then $f \overset{l}{\sim} g$ implies $f \circ k \overset{l}{\sim} g \circ k$.
- If $h, k : Z \to X$ and $f : X \to Y$ then $h \overset{l}{\sim} k$ implies $f \circ h \overset{l}{\sim} f \circ k$.

Then for any $f, g : X \to Y$ and $h, k : Z \to X$ we have $f \overset{l}{\sim} g$ and $h \overset{l}{\sim} k$ implies

$$f \circ h \overset{l}{\sim} g \circ h \overset{l}{\sim} g \circ k.$$

The second assertion is fairly evident. If $H : C(Z) \to X$ is a left homotopy from h to k then $f \circ H : C(Z) \to Y$ is a left homotopy from $f \circ h$ to $f \circ k$.

To prove the first assertion, choose (by (1) of this proposition) a very good cylinder object $C(X)$ and a left homotopy $H : C(X) \to Y$ from $f \to g$. Choose also a good cylinder object $C(Z)$ for Z. Then we have a diagram

$$\begin{array}{ccccc} Z\coprod Z & \xrightarrow{k \sqcup k} & X\coprod X & \longrightarrow & C(X) \\ \downarrow & & & & \downarrow \\ C(Z) & \longrightarrow & Z & \xrightarrow{k} & X \end{array}$$

which commutes. The map on the left is in Cof as $C(Z)$ is good, the map on the right is in $Fib \cap \mathcal{W}$ as $C(X)$ is very good, thus we can find a lift $L : C(Z) \to C(X)$. $H \circ L$ is then a left homotopy from $f \circ k$ to $g \circ k$.

7. Well be here a bit later (I got tired typing all this :-( ). $\square$

Corollary 5.17. *There is a diagram of categories*

$$\begin{array}{ccc} \pi\mathcal{M}_c & & \pi\mathcal{M}_f \\ & \nwarrow \quad \nearrow & \\ & \pi\mathcal{M}_{cf} & \end{array}$$

where

1. $\pi\mathcal{M}_c$ *is the category with* $\operatorname{Ob}\pi\mathcal{M}_c$ *be the set of cofibrant objects of* $\mathcal{M}$ *and* $\pi\mathcal{M}_c(X, Y) = \pi^r(X, Y)$.

2. $\pi\mathcal{M}_f$ *is the category with* $\operatorname{Ob}\pi\mathcal{M}_f$ *be the set of fibrant objects of* $\mathcal{M}$ *and* $\pi\mathcal{M}_f(X, Y) = \pi^l(X, Y)$.

3. $\pi\mathcal{M}_{cf}$ *is the category with* $\operatorname{Ob}\pi\mathcal{M}_{cf}$ *be the set of fibrant and cofibrant objects of* $\mathcal{M}$ *and*

$$\pi\mathcal{M}_{cf}(X, Y) = \pi^r(X, Y) = \pi^l(X, Y) = \pi(X, Y).$$

4. *The functors* $\pi\mathcal{M}_{cf} \hookrightarrow \pi\mathcal{M}_c$ *and* $\pi\mathcal{M}_{cf} \hookrightarrow \pi\mathcal{M}_f$ *are the incusion functors.*

Proof. $\pi\mathcal{M}_f$ is a category as we can compose left homotopy classes between fibrant objects by Proposition 5.16 (6) and the identity morphisms are chosen to be the classes $[id_X] \in \pi^l(X, X)$. The dual argument proves that fro $\pi\mathcal{M}_c$. $\square$

5.4 Localization and derived functors

Let $\mathcal{M}$ be a model category with $\mathcal{W}$, Fib and Cof.

Theorem 5.18. *The homotopy category* $\operatorname{Ho}\mathcal{M}$ *of* $\mathcal{M}$ *exists (without going to a higher universe). Its set of objects can be chosen to be equal to* $\operatorname{Ob}\mathcal{M}$*, and there is an equivalence between* $\operatorname{Ho}\mathcal{M}$ *and* $\pi\mathcal{M}_{cf}$.

Proof. For every object X of $\mathcal{M}$ choose a factorization

$$\emptyset \longrightarrow QX \longrightarrow X$$

of the unique map $\emptyset \to X$ in such a way that QX is cofibrant and $\gamma_X : QX \to X$ is a trivial fibration. Moreover, for every cofibrant X set γ_X to be equal to id_X. Also choose, for every $Y \in \mathcal{M}$, a factorization

$$Y \longrightarrow RY \longrightarrow *$$

of the unique map $Y \to *$ in such a way that RY is fibrant and $i_Y : Y \to RY$ is a trivial cofibration. Moreover, for every fibrant Y set i_Y to be equal to id_Y.

For every morphism $f : X \to Y$ we have a diagram

$$\begin{array}{ccccc} & & & & QY \\ & & & & \downarrow \\ QX & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

By **CM3** there is a lift $Q(f) : QX \to QY$. Moreover, two such lifts are left homotopic by Proposition 5.16 (5). This implies that $Q(g \circ f) \overset{l}{\sim} Q(g) \circ Q(f)$ and $Q(id_X) \overset{l}{\sim} id_{QX}$. Proposition 5.16 (4) left homotopy on cofibrant objects implies right homotopy, so we defined a functor $Q : \mathcal{M} \to \pi\mathcal{M}_c$. In the dual way, the assignment $X \mapsto RX$ defines a functor $R : \mathcal{M} \to \pi\mathcal{M}_f$.

Let QX, QY be as above and $i_{QX} : QX \to RQX$ and $i_{QY} : QY \to RQY$ are trivial cofibrations chosen for every object of $\mathcal{M}$ (by the procedure described above). If $f, g : QX \to QY$ are two right homotopic maps, the dual of Proposition 5.16 (6) implies that $i_{QY} \circ f$ and $I_{QX} \circ g$ are right homotopic. We then observe that, after we extend f and g to $Rf, Rg : RQX \to RQY$, the dual of Proposition 5.16 (5) implies that Rf and Rg are (right) homotopic as well. Thus the assignment $X \mapsto RX$ defines a functor $R : \pi\mathcal{M}_c \to \pi\mathcal{M}_{cf}$. This in total gives us a functor $RQ : \mathcal{M} \to \pi\mathcal{M}_{cf}$.

We define $\mathrm{Ho}\,\mathcal{M}$ as follows. $\mathrm{Ob}\,\mathrm{Ho}\,\mathcal{M} = \mathrm{Ob}\,\mathcal{M}$. $\mathrm{Ho}\,\mathcal{M}(X, Y) = \pi(RQ(X), RQ(Y))$. We denote by $p : \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ the functor which is identity on the objects and is given on morphisms by RQ. By Proposition 5.16 (7), this functor takes $\mathcal{W}$ to isomorphisms. The only remark to be made here is that if $f : X \to Y$ is in $\mathcal{W}$ then a lift Qf and an extension Rf are weak equivalences due to **CM1**.

We now need to check that any functor $F : \mathcal{M} \to \mathcal{D}$ taking $\mathcal{W}$ to $Iso_{\mathcal{D}}$ factors through $\mathrm{Ho}\,\mathcal{M}$ in a unique way. That is, we construct $\bar{F} : \mathrm{Ho}\,\mathcal{M} \to \mathcal{D}$ such that $\bar{F} \circ p = F$. On objects, $\bar{F}$ is clear to define.

On morphisms, first note that F identifies (left or right) homotopic maps. For example, if $f, g : X \to Y$ are left homotopic, choose a homotopy $H : C(X) \to Y$. The map $q : C(X) \to X$ is a weak equivalence, so $F(q)$ is an isomorphism. As the inclusions $i_0, i_1 : X \to C(X)$ satisfy $q \circ i_0 = q \circ i_1$, we obtain $F(i_0) = F(i_1)$. This implies that $F(f) = F(H \circ i_0) = F(H \circ i_1) = F(g)$. The proof for right homotopy is dual.

Next, any element $[f]$ of $\mathrm{Ho}\,\mathcal{M}(X, Y)$ can be represented by a morphism $\tilde{f} : RQX \to RQY$. There is a diagram

$$\begin{array}{ccc} RQX & \xrightarrow{\tilde{f}} & RQY \\ \uparrow & & \uparrow \\ QX & & QY \\ \downarrow & & \downarrow \\ X & & Y \end{array}$$

where the column maps are weak equivalences, so F takes them to isomorphisms. We then can define $\bar{F}([f])$ as the composite

$$F(X) \xleftarrow{\sim} F(QX) \xrightarrow{\sim} F(RQX) \xrightarrow{F(\tilde{f})} F(RQY) \xleftarrow{\sim} F(QY) \xrightarrow{\sim} F(Y).$$

As F identifies homotopic maps, this is independent of the choice of $\tilde{f}$ in $[f]$. All this leads us to observation that $\bar{F}$ is a well defined functor. We also see that $\bar{F} \circ p = F$ by construction[22]. □

Corollary 5.19. *If $p : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ is a localization functor and $f : X \to Y$ is a morphism in $\mathcal{M}$ such that $p(f)$ is an isomorphism, then f is a weak equivalence. That is, $\mathcal{M}$ is* saturated.

Proof. This follows from the previous Theorem and Proposition 5.16 (7). □

We now turn to derived functors.

Theorem 5.20. *Let $F : \mathcal{M} \to \mathcal{D}$ be a functor which takes weak equivalences between cofibrant objects to isomorphisms. Then absolute left derived functor $\mathbb{L}F$ of F exists. Dually, if F takes weak equivalences between fibrant objects to isomorphisms, then the right derived functor $\mathbb{R}F$ of F exists.*

Proof. We first note that if Y is cofibrant and

$$Y \xrightarrow{i} P(Y) \xrightarrow{(p_0, p_1)} Y \times Y$$

is a *very good* path object then, as i is a cofibration and a weak equivalence, we observe that $F(i)$ is an isomorphism as $P(Y)$ is also cofibrant in this case. It allows us to use the same argument as in the previous proof to show that if f and g are right homotopic maps between cofibrant X and Y, then $F(f) = F(g)$.

Consequently, F induces a functor $F^\pi : \pi\mathcal{M}_c \to \mathcal{D}$. We can compose the functor $Q : \mathcal{M} \to \pi\mathcal{M}_c$ (which was constructed in the proof of the previous theorem) with F^π and obtain a functor $F^\pi \circ Q$ which takes $\mathcal{W}$ to isomorphisms (remember that Q lifts weak equivalences to weak equivalences between cofibrant objects). Thus there is a functor $\mathbb{L}F : \operatorname{Ho}\mathcal{M} \to \mathcal{D}$ such that $\mathbb{L}F \circ p = F^\pi \circ Q$.

For every object X we have a morphism $\gamma_X : QX \to X$ which is an element of $Fib \cap \mathcal{W}$ and is identity whenever X is cofibrant. We can apply F to γ_X. As $F^\pi \circ Q(X)$ equals $F(QX)$, we obtain a morphism $F(\gamma_X) : \mathbb{L}F \circ p(X) \to F(X)$. Thus we define α_X to be $F(\gamma_X)$. It is checked to be a natural transformation; moreover, α_{QX} is seen to be the identity map due to the choice of γ_X for cofibrant objects.

[22] The reader might want to explicitly explain how p constructs a class in $\pi(RQX, RQY)$ out of a morphism in $\mathcal{M}(X, Y)$ to see this work.

Given a natural transformation $\beta : G \circ p \to F$ we need to show that there is unique universal $\bar{\beta} : G \to \mathbb{L}F$. If such a $\bar{\beta}$ exists, then we can draw the following commutative diagram[23]

$$\begin{array}{ccccc}
G(X) & \xrightarrow{\bar{\beta}_X} & \mathbb{L}F(X) & \xrightarrow{\alpha_X = F(\gamma_X)} & F(X) \\
\uparrow {\scriptstyle G \circ p(\gamma_X)} & & \uparrow {\scriptstyle \mathbb{L}F \circ p(\gamma_X) = id_{F(QX)}} & & \uparrow {\scriptstyle F(\gamma_X)} \\
G(QX) & \xrightarrow{\bar{\beta}_{QX}} & \mathbb{L}F(QX) & \xrightarrow{\alpha_{QX} = id_{F(QX)}} & F(QX)
\end{array}$$

As γ_X is a weak equivalence, $G \circ p(\gamma_X)$ admits an inverse. We thus see that β_X must be equal

$$\begin{array}{ccc}
G(X) & & \mathbb{L}F(X) \\
\downarrow {\scriptstyle (G \circ p(\gamma_X))^{-1}} & & \uparrow {\scriptstyle \mathbb{L}F \circ p(\gamma_X) = id_{F(QX)}} \\
G(QX) & \xrightarrow{\beta_{QX}} & \mathbb{L}F(QX)
\end{array}$$

but, as $\alpha_Q X : \mathbb{L}F(QX) \to F(QX)$ is the identity, $\bar{\beta}_{QX}$ is the same morphism as $\beta_{QX} : G(QX) \to F(QX) = \mathbb{L}F(QX)$. In other words, the map $\bar{\beta}$ is uniquely defined on cofibrant objects by β, and is uniquely extended to non-cofibrant objects by the diagram above.

The fact that $\mathbb{L}F$ is absolute is checked easily as we have presented an explicit construction of a left derived functor for F, and for any $G : \mathcal{D} \to \mathcal{E}$ the functor $G \circ F$ satisfies the assumption of the theorem. $\square$

Example 5.21. For a (commutative, for simplicity) ring A consider the category $\mathbf{DGMod}_A^{\leq 0}$ with projective model structure. For every module $M \in \mathbf{Mod}_A$ there is a functor

$$- \otimes_A M : \mathbf{DGMod}_A^{\leq 0} \to \mathbf{DGMod}_A^{\leq 0}$$

which takes a complex

$$... \to N^i \to N^{i+1} \to ...$$

to

$$... \to N^i \otimes_A M \to N^{i+1} \otimes_A M \to ...$$

This functor does not, in general, take quasiisomorphisms to quasiisomorphisms, but this property

[23] We use the fact that $p : \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ is identity on the objects.

is fulfilled when this functor is restricted to complexes of projective modules. This basically follows from the fact that every short exact sequence of projective modules

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

is split, $P_2 \cong P_1 + P_3$.

Now, every cofibrant object of $\mathbf{DGMod}_A^{\leq 0}$ is a complex of projective modules, so $- \otimes_A M$ has a left derived functor

$$- \otimes_A^{\mathbb{L}} M : \mathrm{Ho}\,(\mathbf{DGMod}_A^{\leq 0}) \to \mathrm{Ho}\,(\mathbf{DGMod}_A^{\leq 0}).$$

In fact, there is a bit more to tell about this example. In classical notation, $\mathrm{Ho}\,(\mathbf{DGMod}_A^{\leq 0})$ is denoted as $D^-(A)$ and is called *the right-bounded derived category of the ring A*. We have proved that this category is equivalent to the category of fibrant-cofibrant objects of $\mathbf{DGMod}_A^{\leq 0}$ with homotopy classes as maps.

Let $M^\bullet$ be a complex. Define $C(M^\bullet)^\bullet$ to be the complex with

$$C(M^\bullet)^i = M^i \oplus M^{i+1} \oplus M^i$$

with differential

$$d^i : (x, a, y) \mapsto (d^i x + a, -d^{i+1} a, d^i y + a).$$

There are two obvious maps

$$M^\bullet \oplus M^\bullet \to C(M^\bullet)^\bullet,\ (x, y) \mapsto (x, 0, y)$$

and

$$C(M^\bullet)^\bullet \to M^\bullet,\ (x, a, y) \mapsto x + y$$

(which factor the codiagonal) and the latter one is checked to be a quasiisomorphism. Thus we see that $C(M^\bullet)^\bullet$ is a cylinder object for $M^\bullet$. Moreover, one can explicitly check that left homotopy corresponds to the notion of chain homotopy from homological algebra. Thus we obtained the well-known result that $D^-(A)$ is equivalent to the category $K^-(Proj{-}A)$, the category of (right-bounded) complexes of projective modules with morphisms chain maps modulo chain homotopies.

6 Examples

6.1 Cofibrantly generated model categories

6.1.1 Small, or finitely presented, objects

Let $\mathcal{C}$ be a category.

Definition 6.1. A partially ordered set I with order relation $\leq$ is called *directed* if for any two i, j in I there is an element $k \in I$ such that $i \leq k, j \leq k$.

That is, if we consider I as a category with at most one morphism between two objects, the condition above says that for any two objects i and i there always exist two morphisms $f_{ik} : i \to k$ and $f_{jk} : j \to k$ with common codomain k.

Definition 6.2. A *directed colimit* is a colimit of a functor $F : I \to \mathcal{C}$, where I is a directed poset (viewed as a category). Such functor is called a *directed diagram* in $\mathcal{C}$.

Example 6.3. In **Set**, if $F : I \to \mathbf{Set}$ is a directed diagram, we can construct its colimit in the following way. Take $\coprod_{i \in \operatorname{Ob} I} F(i)$ and consider the following relation: $x \in F(i) \sim y \in F(j)$ if and only if there is $k \in I$, $f_{ik} : i \to k$ and $f_{jk} : j \to k$ such that $F(f_{ik})(x) = F(f_{jk})(y)$. This is an equivalence relation. One can check now that $\varinjlim F$ can be identified with $\coprod_{i \in \operatorname{Ob} I} F(i)/\sim$.

This construction applies to many categories structured over sets, e.g. modules and rings, and can be used to show that the forgetful functor from these categories to **Set** preserves directed colimits.

Example 6.4. Let S be a set. Then there is a poset $P(S)$ of all subsets of S ordered by inclusion. It is seen to be directed, as the union of two subsets is a subset. This poset has a terminal object, so colimits over it are uninteresting. But let S be infinite and consider the poset $P(S)_{fin}$ of all finite subsets, ordered by inclusion, which is also directed.

Now, there is a 'canonical' diagram $P(S)_{fin} \to \mathbf{Set}$, sending a subset $x \subset S$ to x. The reader can check that the colimit over this diagram is S. This fact can be summarised as follows: every set if a directed colimit of finite sets.

A similar argument applies to other categories structured over **Set**. For example, every module over a ring A is a directed colimit of finitely presented modules (that is, modules M such that there exists an exact sequence

$$A^m \to A^n \to M \to 0$$

with $n, m \in \mathbb{N}$).

Thus we see that directed colimits correspond to 'unions of subobjects'. In the example above, however, these subobjects were finite, or small, in some sense. We are now going to formalize this.

Definition 6.5. An object X of $\mathcal{C}$ is called *finitely presented* if, for any directed diagram $F : I \to \mathcal{C}$ (such that $\varinjlim F$ exists), the canonical morphism

$$\varinjlim \mathcal{C}(X, F(-)) \to \mathcal{C}(X, \varinjlim F)$$

is an isomorphism. Here $\mathcal{C}(X, F(-))$ is the functor from i to **Set**, $i \mapsto \mathcal{C}(X, F(i))$.

Remark 6.6. The set $\varinjlim \mathcal{C}(X, F(-))$ is, as we have seen in Example 6.3, a quotient of $\coprod_{i \in I} \mathcal{C}(X, F(i))$. One can consequently check the following: the canonical morphism

$$\varinjlim \mathcal{C}(X, F(-)) \to \mathcal{C}(X, \varinjlim F)$$

is an isomorphism if and only if any morphism $f : X \to \varinjlim F$ factors as

$$\begin{array}{ccc} & F(i) & \\ {}^{\bar{f}}\nearrow & & \searrow \\ X & \xrightarrow{\quad f \quad} & \varinjlim F \end{array}$$

for some $i \in I$ (here $F(i) \to \varinjlim F$ is the canonical morphism).

Example 6.7. The remark above allows one to deduce the following.

SET $X \in \mathbf{Set}$ is finitely presented iff X is finite set.

SSET $X \in \mathbf{SSet}$ is finitely presented iff $X([n])$ is finite for every $[n] \in \Delta$.

MOD $M \in \mathbf{Mod}_A$ is finitely presented iff M is finitely presented in the sense of Example 6.4.

DG $M^\bullet \in \mathbf{DGMod}_A$ is finitely presented iff every M^i is finitely presented and $M^i = 0$ for almost all $i \in \mathbb{Z}$.

TOP For **Top**, the situation with finitely presented objects is subtle. There is a theorem, which says that a finite CW-complex is finitely presented with respect to a certain class of directed colimits: given a sequence of inclusions

$$X_0 \to X_1 \to ... \to X_n \to ...$$

such that, for every $i \geq 0$, every inclusion $X_i \to X_{i+1}$ makes (X_{i+1}, X_i) in a relative CW-pair, any map from a finite CW-complex A to $\varinjlim X_i$ factors through X_k for some k.

CAT A category $\mathcal{D}$ is finitely presented iff the sets $\operatorname{Ob}\mathcal{D}$ and $\operatorname{Mor}\mathcal{D}$ are finite.

6.1.2 Saturated classes of maps and the small object argument

Another story of this section concerns sets of the form $l(I)$ for some set of maps $I \subset \operatorname{Mor}\mathcal{C}$ of a category $\mathcal{C}$.

We have seen that fibrations in a model category $\mathcal{M}$ often form a set $r(J)$ for some small set $J \subset \operatorname{Mor}\mathcal{M}$. Cofibrations then are determined as $Cof = l(Fib \cap \mathcal{W})$, and trivial cofibrations are determined as $Cof \cap \mathcal{W} = l(Fib) = l(r(J))$. It turns out that in many examples trivial fibrations also satisfy $Fib \cap \mathcal{W} = r(I)$ for some small $I \subset \operatorname{Mor}\mathcal{M}$. Consequently, $Cof = l(r(I))$.

Example 6.8. One can prove the following facts.

TOP In **Top**, it can be checked that, in the notation as above, I is the set of all maps $S^{n-1} \to D^n$ for $n \geq 0$ ($S^{-1} = \emptyset$).

SSET In **SSet**, I is the set of all inclusions $\partial\Delta^n \to \Delta^n$ for $n \geq 0$.

DG In $\mathbf{DGMod}_A$, I is the set of all maps $S_{n+1} \to D_n$, where S^n is the complex with $S_n^i = A$ if $i = n$ and 0 otherwise, and D_n is the complex with $D_n^i = A$ when $i = n$ or $n+1$ and zero otherwise (the differential $D_n^n \to D_n^{n+1}$ is the identity map).

CAT In **Cat**, consider the subcategory $\partial[1]$ of $[1]$ which does not contain the morphism $0 \to 1$. Also consider the category $P = [1] \coprod_{\partial[1]} [1]$ which is easily seen to be a category with objects $0, 1$ and two distinct morphisms $0 \to 1$ (the 'parallel-arrow' category). There are evindent injections $u : \emptyset \to [0]$, $v : \partial[1] \to [1]$, and a projection $w : P \to [1]$ (v and w are identities on the sets of objects). Then one can show that $I = \{u, v, w\}$.

Let us delve a bit more into the structure of $l(r(I))$.

Definition 6.9. A set of morphisms $S \subset \operatorname{Mor} \mathcal{C}$ of a cocomplete category $\mathcal{C}$ is called *saturated*[24] if and only if the following properties hold.

1. S is closed under pushouts of coproducts: given a set L and, for any $l \in L$, a morphism $f_l : C_l \to D_l$ in S, for $X \in \mathcal{C}$ and any pushout diagram

$$\begin{array}{ccc} \coprod_{l \in L} C_l & \longrightarrow & X \\ {\scriptstyle \coprod_{l \in L} f_l} \downarrow & & \downarrow {\scriptstyle g} \\ \coprod_{l \in L} D_l & \longrightarrow & Y \end{array}$$

the induced morphism $g : X \to Y$ is in S.

2. S is closed under finite and countable compositions. The latter means that, given a diagram[25] $X : \mathbb{N} \to \mathcal{C}$ such that, for any $i \in \mathbb{N}$, the map $X(i) \to X(i+1)$ belongs to S, we always have that the induced map $X(0) \to \varinjlim X$ belongs to S.

3. S is closed under retracts (see Definition 5.1).

Proposition 6.10. *For any $I \subset \operatorname{Mor} \mathcal{C}$, $l(r(I))$ is saturated.*

Proof. Half of things have been checked back in Proposition . We only note that a coproduct of $f_l : C_l \to D_l \in l(r(I))$ is again in $l(r(I))$: just choose a lift separately for each f_l. The stability under countable compositions happens for the same reason: lifts can be chosen, in this case, inductively. □

One can see that an intersection of a collection of saturated sets of morphisms is again saturated; consequently, for $I \subset \operatorname{Mor} \mathcal{C}$ we can talk about $\overline{I}$, the minimal saturated set of morphisms containing I.

When every morphism $f : C \to D$ in I has finitely presented domain C, interesting things start to occur.

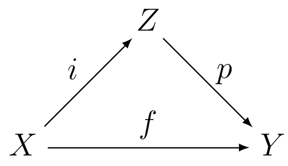
[24] We are ignoring in this section the cases of α-saturation, where α is a regular cardinal, and restricting ourselves to the case of $\alpha = \omega$, the first infinite cardinal.

[25] We view the totally ordered set $\mathbb{N}$ here as a category.

Theorem 6.11 (The small object argument). *Let $\mathcal{C}$ be a cocomplete category with $I \subset \operatorname{Mor} \mathcal{C}$ such that for every $C \to D \in I$, the object C is finitely presented. Then there exists a functor*[26]

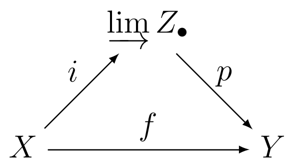
$$T : \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$$

which sends a morphism $f : X \to Y$ to a 2-simplex



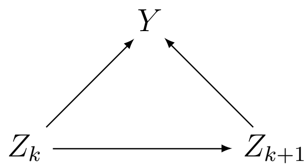
where $p \in r(I)$ and $i \in \overline{I}$.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. We construct a functor $Z_\bullet : \mathbb{N} \to \mathcal{C}$ such that $Z_0 = X$, and a morphism $Z_\bullet \to \Delta_{\mathbb{N}} Y$, where $\Delta_{\mathbb{N}} Y$ is the constant functor $\mathbb{N} \to \mathcal{C}$ with value Y. This is the same as a morphism $p : \varinjlim Z_\bullet \to Y$. We then factor f as



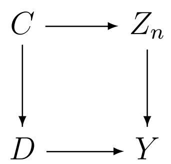
where $i : X = Z_0 \to \varinjlim Z_\bullet$ is the natural inclusion. We then show that i and p have desired properties.

Proceed by induction. Assume that Z_k and maps between them have been defined for $k \leq n$. also assume that the maps $Z_k \to Y$ have been constructed for $k \leq n$ so that triangles



commute (the assumption is saitsfied for $n = 0$). We now have to construct Z_{n+1} and two maps $Z_n \to Z_{n+1}$, $Z_{n+1} \to Y$.

Consider a commutative diagram



for $C \to D$ in I and $Z_n \to Y$ is the map from the inductive assumption. Let S_n denote the set of all such commutative diagrams. We observe that there is a functor from (discrete) category S_n to $\mathcal{C}^{[1]}$,

[26]We remind the reader that for two categories A and B, B^A denotes the category $Fun(A, B)$.

mapping the diagram

$$\begin{array}{ccc} C & \longrightarrow & Z_n \\ \downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

to $C \to D$. A colimit over this diagram is a coproduct[27]

$$\coprod_{S_n} C \to \coprod_{S_n} D$$

Moreover, as every map in this coproduct was supplied with a morphism in $\mathcal{C}^{[1]}$ to $Z_n \to Y$, we finally get a commutative diagram

$$\begin{array}{ccc} \coprod_{S_n} C & \longrightarrow & Z_n \\ \downarrow & & \downarrow \\ \coprod_{S_n} D & \longrightarrow & Y \end{array}$$

We now define Z_{n+1} to be the colimit of

$$\begin{array}{ccc} \coprod_{S_n} C & \longrightarrow & Z_n \\ \downarrow & & \\ \coprod_{S_n} D & & \end{array}$$

The universal property of colimit tells us that there is a map $Z_{n+1} \to Y$ such that

$$\begin{array}{ccccc} \coprod_{S_n} C & \longrightarrow & Z_n & & \\ \downarrow & & \downarrow & \searrow & \\ \coprod_{S_n} D & \longrightarrow & Z_{n+1} & \longrightarrow & Y \end{array}$$

commutes. The choices of maps $Z_n \to Z_{n+1}$ and $Z_{n+1} \to Y$ are evident from this diagram.

The class $\overline{I}$ is saturated. By construction we observe that every $Z_k \to Z_{k+1}$ belongs to $\overline{I}$, thus $i : X = Z_0 \to \varinjlim Z_\bullet$ belongs to $\overline{I}$.

It remains to show that $p \in r(I)$. Let

$$\begin{array}{ccc} C & \longrightarrow & \varinjlim Z_\bullet \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle p} \\ D & \longrightarrow & Y \end{array}$$

[27]This can be a coproduct over a huge set even if the original set I is finite.

be a commutative diagram with $\alpha : C \to D$ in I. As C is finitely presented, the morphism $C \to \varinjlim Z_\bullet$ in this diagram factors through Z_n for some n. Thus we are left with a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & \varinjlim Z_n \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

But this diagram is an element x of S_n and consequently we can draw the following diagram

$$\begin{array}{ccccc} C & \longrightarrow & \coprod_{S_n} C & \longrightarrow & Z_n \\ {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & \coprod_{S_n} D & \longrightarrow & Z_{n+1} \end{array}$$

Here the left square maps α to its summand in the coproduct as dictated by x. The composite of the down row gives us a map $D \to Z_{n+1}$. Composing with the canonical $Z_{n+1} \to \varinjlim Z_\bullet$, we obtain a map $D \to \varinjlim Z_\bullet$, which is checked to be the required lift. □

Corollary 6.12. *In the conditions of the theorem above,* $\overline{I} = l(r(I))$.

Proof. It is clear that $\overline{I} \subset l(r(I))$. For the converse, suppose that $f \in l(r(I))$. Factor f as $f = p \circ i$, where $i \in \overline{I}$ and $p \in r(I)$. The retract argument (Proposition 5.8) then shows that f is a retract of i, and so $f \in \overline{I}$.

Remark 6.13. We observe that $\overline{I} = l(r(I))$ consists of morphisms which are retracts of countable compositions of sequences

$$X_0 \to X_1 \to X_2 \to ...$$

such that, for $i \geq 0$, there are pushout squares

$$\begin{array}{ccc} \coprod_{S_i} C & \longrightarrow & Z_i \\ \downarrow & & \downarrow \\ \coprod_{S_i} D & \longrightarrow & Z_{i+1} \end{array}$$

for some set S_i. We may denote the set of all such countable compositions as $\mathbf{cell}(I)$. Thus $l(r(I))$ consists of retracts of elements of $\mathbf{cell}(I)$.

6.1.3 Cofibrantly generated model categories

The results above suggest that, if, in a model category $\mathcal{M}$ we can choose sets I and J of cofibrations and trivial cofibrations, then, if domains of arrows in I and J are finitely presented, then the axiom **CM4** is automatically satisfied provided that $r(I) = Fib \cap \mathcal{W}$ and $r(J) = Cof$. This is formalised as follows.

Definition 6.14. A model category $\mathcal{M}$ is called *cofibrantly generated* if there exist sets $I \subset Cof$ and $J \subset Cof \cap \mathcal{W}$, called *generating cofibrations* and *generating trivial cofibrations* such that

1. Every morphism $f : C \to D$ in I or in J has the property that its domain C is finitely presented.
2. $Fib = r(J)$ and $Fib \cap \mathcal{W} = r(I)$.

This can be turned the other way, into the 'Recognition Lemma' of D.M. Kan.

Proposition 6.15. *Let $\mathcal{M}$ be a complete and cocomplete category with $\mathcal{W}, I, J \subset \mathrm{Mor}\,\mathcal{M}$. Assume that the following holds:*

1. *$\mathcal{W}$ satisfies 3-for-2 and is closed under retracts.*
2. *Every morphism $f : C \to D$ in I or in J has the property that its domain C is finitely presented.*
3. *$l(r(J)) \subseteq l(r(I)) \cap \mathcal{W}$.*
4. *$r(I) \subseteq r(J) \cap \mathcal{W}$.*
5. *At least one of the inclusions above is, in fact, a bijection.*

Then $\mathcal{M}$ admits a model structure with weak equivalences $\mathcal{W}$, $Fib = r(J)$ and $Cof = l(r(I)) = \overline{I}$. One also has in this case $Fib \cap \mathcal{W} = r(I)$ and $Cof \cap \mathcal{W} = l(r(J)) = \overline{J}$.

Proof. **CM0** and **CM1** are valid by assumption (1). **CM2** also follows from the assumption (1) on $\mathcal{W}$ and the fact that $l(r(I))$ and $r(J)$ are stable under retracts. **CM4** follows from the small object argument, which works due to the assumption (2), and also due to, (3), (4), which allow us to observe that $\overline{J}$ are among trivial cofibrations and $r(I)$ are among trivial fibrations.

For **CM3**, assume that $l(r(J)) = l(r(I)) \cap \mathcal{W}$ (the other case is similar). Then 'trivial cofibration - fibration' part of **CM3** is trivial. Now let $f \in r(J) \cap \mathcal{W}$, that is, a trivial fibration. Factor $f = p \circ i$, where $p \in r(I)$ and $i \in \overline{I}$. $\mathcal{W}$ satisfies 3-for-2, so, as $p \in r(I) \subseteq r(J) \cap \mathcal{W}$, we get that $i \in l(r(I)) \cap \mathcal{W} = l(r(J))$. We now can use the retract argument (Proposition 5.8) to see that f is a retract of p, hence $r(I) = r(J) \cap \mathcal{W}$ and we observe that **CM3** is satisfied. □

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