Problems

April 12, 2012

1. Describe explicitly (co)products and (co)equalizers in **Set**, **Top**, $\mathbf{Mod}_A$, **SSet** and $\mathbf{DGMod}_A$. Explain how to compute a (co)limit of an arbitrary small diagram in all these categories.

2. For $X \in \mathbf{SSet}$ denote by $sk_n(X)$ the minimal simplicial subset of X containing all simplices of X of degree n and less (put $sk_{-1}(X) = \emptyset$).

   - Prove that $sk_n(X)$ is isomorphic to the colimit of the following functor $F : (\Delta_X^{\leq n})_N \to \mathbf{SSet}$. Here $(\Delta_X^{\leq n})_N$ is the subcategory of the category of elements Δ_X of X (see lectures) containing all non-degenerate simplices of degree n and less, and F sends a non-degenerate simplex $\Delta^k \to X$ to $\Delta^k \in \mathbf{SSet}$.
   - Prove that for any $n \in \mathbb{N}$ there is a pushout diagram

   $$\begin{array}{ccc} \coprod \partial\Delta^n & \longrightarrow & sk_{n-1}(X) \\ \downarrow & & \downarrow \\ \coprod \Delta^n & \longrightarrow & sk_n(X) \end{array}$$

   What is the set over which the coproduct on the left is taken?
   - Prove that the realisation $|\partial\Delta^n|$ is isomorphic to S^n in **Top**.
   - Prove that $|X|$ is a CW-complex.

3. In Δ, define[1] $\partial_i : [n-1] \to [n]$ to be the unique injective monotone map not containing i in its image. Denote also by $\sigma_i : [n+1] \to [n]$ the unique surjective monotone map which maps i and $i+1$ in $[n+1]$ to i in $[n]$. For a simplicial set X, denote by $d_i = X(\partial_i) : X(n) \to X(n-1)$ and $s_i = X(\sigma_i) : X(n) \to X(n+1)$. These are called, respectively, i-th *face* and *degeneracy* maps.

   - Prove the identities

   $$d_i d_j = d_{j-1} d_i \, (i < j),$$

   $$d_i s_j = d_{j-1} s_i \, (i < j),$$

   $$d_j s_j = 1 = d_{j+1} s_j$$

   $$d_i s_j = s_j d_{i-1} \, (i > j+1)$$

   $$s_i s_j = s_{j+1} s_i \, (i \leq j)$$

[1] $[n]$ is omitted from the notation

- Prove that any injective (surjective) map in Δ can be written as a composition of ∂_i (σ_i). Consequently, prove that any map in Δ can be written as a composition of σ_i followed by ∂_i.
- Prove that $\partial\Delta^n$ is isomorphic to the colimit of

$$\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{0\le i\le n} \Delta^{n-1}$$

(what are the two maps in this diagram?)

- Let $A[-] : \mathbf{Set} \to \mathbf{Mod}_A$ denote the free A-module functor. For a simplicial set X, define $A[X]^{-i} = A[X(i)]$ and

$$d_X^{-i} : A[X(i)] \to A[X(i-1)]$$

to be the sum $\sum_j (-1)^j d_j$. Prove that this gives a functor from $\mathbf{SSet}$ to $\mathbf{DGMod}_A^{\le 0}$.

4. A *groupoid* is a category $\mathcal{C}$ such that any morphism in $\operatorname{Mor}\mathcal{C}$ is an isomorphism

   - Prove that for any category $\mathcal{D}$ and X in $\mathbf{SSet}$, a map $f : X \to N(\mathcal{D})$ is determined by f_0, f_1 and f_2 ($f_n : X(n) \to N(\mathcal{D})(n)$).
   - Prove that for a groupoid $\mathcal{C}$, the nerve $N(\mathcal{C})$ is fibrant in the standard model structure on $\mathbf{SSet}$.

5. Let $\mathcal{M}$ be a model category. Prove *Whitehead's theorem* A morphism $f : X \to Y$ between fibrant-cofibrant objects is a weak equivalence if and only if it is an isomorphism in $\pi\mathcal{M}_{cf}$. (Hint: for 'only if' part, it is enough to prove it only for trivial (co)fibrations. For 'if' part, it might be useful to factor $f = p \circ i$, so that i is a trivial cofibration and p is a fibration, and then try to show that p is a weak equivalence.)

6. Prove *Ken Brown's lemma*: let $\mathcal{M}$ be a model category. If a functor $F : \mathcal{M} \to \mathcal{D}$ takes trivial cofibrations between cofibrant objects to isomorphisms in $\mathcal{D}$, then F takes all weak equivalences between cofibrant objects to isomorphisms in $\mathcal{D}$ and the left derived functor $\mathbb{L}F$ exists. (Hint: if $f : A \to B$ is a weak equivalence between cofibrant objects, factor the induced map $A \coprod B \xrightarrow{f \sqcup id_B} B$ as a cofibration followed by trivial fibration. Also, remember that cofibrations are stable under pushout.)

7. Let

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be a Quillen adjunction. Prove that the following are equivalent

   - The pair

$$\operatorname{Ho}(\mathcal{M}) \underset{\mathbb{R}G}{\overset{\mathbb{L}F}{\rightleftarrows}} \operatorname{Ho}(\mathcal{N})$$

   are inverse to each other up to natural isomorphism (that is, $(\mathbb{L}F, \mathbb{R}G)$ is an adjoint equivalence of categories).

   - For any cofibrant object X of $\mathcal{M}$ and any fibrant object Y of $\mathcal{N}$ a morphism $F(X) \to Y$ is a weak equivalence in $\mathcal{N}$ if and only if the adjoint morphism $X \to G(Y)$ is a weak equivalence in $\mathcal{M}$.