Networks and Spatial Economics, 3: (2003) 371–395

Stationary Dynamic Solutions in Congested Transportation Networks: Summary and Perspectives

YURII NESTEROV
CORE, Université Catholique de Louvain, Belgium

ANDRÉ DE PALMA
Institut Universitaire de France, THEMA, Université de Cergy-Pontoise, France
email: depalma@eco.u-cergy.fr

Abstract

The objective of this paper is to provide a systematic treatment of the stationary equilibrium dynamic solutions in large congested networks. We describe the Stable Dynamics approach that is based only on logical assumptions, and is amenable to a rigorous mathematical description. All parameters in use have a direct physical meaning and interpretation. We show existence of the stationary solutions under fairly weak assumptions. For completeness, we present (and criticize) the standard static traffic assignment models and discuss the key differences.

Keywords: Stable dynamics, static traffic assignment, congestion, variational inequalities

1. Introduction

We start this paper by a short historical overview of the models used for equilibrium traffic assignment. Traditionally, in Transportation Science there exists two extreme types of network assignment models: *static* network assignment models and *dynamic* network assignment models.

Static network assignment models have been developed since about half a century. The first mathematical description of those models is due to Beckmann et al. (1956). Static network assignment models are described as follows: given a network, congestion laws and an origin-destination (O-D) matrix, find a user- equilibrium regime. The user-equilibrium (UE) concept is based on the (first) Wardrop principle (1952), which essentially says that at equilibrium each driver selects the shortest route. In their seminal formulation, Beckmann et al. show that their problem can be formulated as a convex optimization program (with a non-linear objective function and linear constraints). The output of their model consists in the equilibrium flows on all arcs of the network. The social optimum (SO) solution provides a second benchmark. At the social optimum, the total cost, sum of all user costs is minimized (second Wardrop Principle). This second benchmark corresponds to an ideal since the decentralization of the social optimum (e.g. via road pricing) is not feasible from a practical point of view. In other words, there exists no set of realistic policies that could be implemented in such a way that all congestion externalities are internalized (see Lévy-Lambert, 1968), and all the subsequent literature in Transportation Economics (see also the discussion on marginal cost pricing, discussed by W. Vickrey, in particular in the context of public transportation (Arnott et al., 1994)).

During the last decades, static network assignment models have been extended in many directions. Below, we very briefly mention the major extensions of static models. The variational inequality formulation of static assignment models has been introduced by Dafermos (1972). In the original formulation, the travel time on an arc only depends on the flow entering this arc. With the variational inequality approach, interaction between flows (to model intersections, for example) is possible. Moreover, when the static network assignment models is formulated as a variational inequalities problem, it can be solved using standard numerical methods. The stochastic user equilibrium (SUE) formulation (in which drivers route choice is stochastic) defined by Daganzo and Sheffi (1977) extends the deterministic case introduced by Beckmann et al. (1956). At the SUE, no driver can decrease his *perceived* travel time by unilaterally changing route (see, Daganzo and Sheffi, 1977 and Sheffi, 1985). In this case, typically drivers select route according to a (Logit or Probit) discrete choice model (see, Ben-Akiva and Lerman, 1985 and Anderson et al., 1992). In the variable or elastic demand extension, the O-D matrix is not fixed and the number of drivers depends on the travel conditions. The mathematical program formulation of this problem was already proposed in Beckmann et al. (1956). The mode choice model, which belongs to this family of extensions, is described in Florian (1977) followed afterwards by numerous work. Finally, the problem with multi-class users takes into account different segments (see, e.g. Dafermos, 1972). A segment can be characterized by specific value of time, specific route choice behavior, specific congestion law, inter alia. Various algorithms were also proposed to solve those problems. Starting with the Frank and Wolf algorithm (1956), later on implemented using various acceleration procedures (see, e.g. LeBlanc et al., 1985; for a more general discussion on numerical procedure, see Chen et al., 2000). However, in order to interpret the output of these models, we need to find a reasonable and possibly intuitive dynamic explanation of the static solutions.

Dynamic congestion phenomena have received more and more attention during the last decades, first at the supply level and more recently at the demand level. In dynamic network assignment models, travel time on an arc depends on the time of the day, and decisions are concerned not only with route (and mode) choice but also with the time of use. We can safely say that dynamic models have emerged as a new paradigm. Despite this fact, the vast majority of planning software used to simulate policies for large transportation systems still belong to the family of static network assignment models. Perhaps, the main reason for such a situation is that static models are amenable to mathematical formulations, they are less data intensive with respect to dynamic models and they can be easily solved by the modern numerical schemes. Moreover, existence and uniqueness remain very difficult to prove for general dynamic models (especially if they includ spill-backs effects, inter alia).

Intuitively, it is clear that a dynamic process is well described by a static model only if some of its parameters are *constant functions of time*. In this case, we expect that there is a stationary regime which could be provided with a static interpretation. However, as we will argue later, it is not possible to find such an interpretation for the standard static models of traffic congestion.

The main goal of this paper is to fill the existing gap between two approaches introduced above by developing an *intermediate* class of models for which the solutions can be rigorously interpreted as the *stationary regimes* of a dynamic processes. Note that such a methodology is quite standard for System Theory (e.g. Lyapunov stability theory in Control Science). Thus, we will try to develop some standard tools that we believe are candidate for the analysis of dynamic transportation models. Note that such tools have already proved their power in many other fields involving optimization on large networks.

In this paper, we develop the class of *Stable Dynamics* models. These models are based on some natural assumptions on drivers behavior. They rely on a minimum set of parameters for the network, which have a clear physical interpretation. We illustrate this approach with simple examples and then provides a formulation which holds for any type of network. We also establish a connection between Stable Dynamics and the minimal cost multicommodity transportation problem with bounded arc capacities. This connection appears to be quite surprising since the solutions of Stable Dynamics model can be associated with the solution of a Wardrop user equilibrium (UE) problem, while the multicommodity flow problem corresponds to a social optimal (SO) setting. Of course, our models captures only some general features of the traffic congestion. However, it seems that for the large networks this level of details is close to the limit of the performance of the numerical schemes.

The paper is organized as follows. In Section 2 we study several simple examples (two routes in parallel, triangle network, Braess network), which help us to develop some intuition on the stationary solutions of the dynamic assignment problems. In the next Section 3, we introduce the notation and present some standard results on the Beckmann model for static traffic assignment. We show that the main element of these models, the arc travel time function, can be given a natural dynamic interpretation. In Section 4 we describe the Stable Dynamics models for a general network and prove the existence results. We show that the solutions of these models can be interpreted as stationary states of a dynamic process. Concluding remarks are relegated to Section 5. In Appendix 1, we discuss an application of the Beckmann model to public transportation and show that in this case the presence of the arc travel time functions is very natural.

2. Logical stationary solutions

Let us start our analysis by finding some logical solutions to several simple examples. We will try to restrict ourselves by the minimal amount of reliable parameters. We assume that, for each arc a of the network we can estimate the minimal free traffic travel time $\bar{t}_a$. The minimum travel time is a function of regulation (but also on driver behavior and enforcements). Let us introduce also another characteristic of the arc, the maximal output flow $\bar{f}_a$. In urban network the maximal flow depends on the number of lanes of the road, the duration of the green light at the intersection, the weather conditions, etc. Surprisingly enough, even from this restricted (and easily available) information we can retrieve the equilibrium travel time for some dynamic flow patterns.

Consider the network $\mathcal{R}$ and the set of O-D pairs. We seek for a stationary regime, during a period of time T. That is, the number of users d_k travelling from node ω_k to δ_k is assumed to depart at an uniform flow $f_k = d_k/T$. For simplicity we assume that $T = 1$, so that $f_k = d_k$.

We now introduce three assumptions. The first one is concerned with the route choice.

Assumption 1. Each driver considers as given the travel time pattern. For each O-D pair, only the minimum travel time routes are selected.

The second assumption is concerned with stationary dynamic solutions. Since we wish to compute a stationary solution, the inflow on a link could never exceed the capacity, that is:

Assumption 2. The flow f_a on arc a never exceeds the capacity $\bar{f}_a$ of this arc: $f_a \leq \bar{f}_a$.

Two situations can occur. Either the flow on arc α is smaller than its capacity; then it is assumed that the travel time on this arc is equal to the minimum travel time. Or the flow is equal to the capacity, in this case we can only guarantee that the travel time is larger or equal to the minimum travel time. Summing up:

Assumption 3. Below capacity the travel time on an arc is equal to the minimum travel time; at capacity it can take any value larger or equal to the minimum travel time, i.e. if $f_\alpha < \bar{f}_a$, then $t_\alpha = \bar{t}_\alpha$ and if $f_\alpha = \bar{f}_a$, then $t_\alpha \geq \bar{t}_\alpha$.

Note that this assumption is not a travel time function; it provides us with a more general object, the *link performance model*.

2.1. *Stationary dynamics for parallel routes*

We consider one origin node ω and one destination node δ connected by two arcs denoted by a and b. The O-D matrix reduces to a single flow, denoted by f. We first assume that f is a constant function of time. Arc i is characterized by the parameters $\{\bar{t}_a, \bar{f}_a\}$, where $\bar{t}_i$ is the minimum travel time, and $\bar{f}_i$ is the capacity of this arc i, $i = a, b$. Without loss of generality, we assume that $\bar{t}_a < \bar{t}_b$. Each driver is then faced with a route choice, either use arc a or use arc b. The analysis could be easily extended to any number of routes in parallel. We first solve for equilibrium.

Three situations occur according to the magnitude of the flow f. We denote by a superscript e the equilibrium values. (In particular, t^e denotes the equilibrium travel time.)

If $f < \bar{f}_a$, all users select arc a, and the equilibrium travel time is given by: $t^e = \bar{t}_a$.

If $\bar{f}_a < f < \bar{f}_a + \bar{f}_b$, all users could not benefit from the shortest arc, a and some users will select arc b. Then $f_a^e = \bar{f}_a$, i.e. the maximum number of users select arc a , while the remaining users select arc b : $f_b^e = f - \bar{f}_a$. By assumption, $f_b^e = f - \bar{f}_a < \bar{f}_b$, so the travel time on arc b is $\bar{t}_b$. Since both arcs are used, they have the same travel time t^e with $t^e = t_a^e = t_b^e = \bar{t}_b$. In particular, note that the equilibrium travel time is a discontinuous function of the total flow f.

If $f = \bar{f}_a$, $f_a^e = \bar{f}_a$ and $f_b^e = 0$, and we cannot assign a specific value to the equilibrium travel time. Any travel time t^e, with $t^e \in [\bar{t}_a, \bar{t}_b]$, is consistent with a stationary dynamic regime.

If $f > \bar{f}_a + \bar{f}_b$ no stable dynamic solution exists since the capacity constraints are necessarily violated.

Let us show that all above constant travel time functions correspond to some dynamic patterns of the demand flow.

Let us check what happens with our solutions in dynamics. Assume that the flows in our network change in time τ. Denote by $f(\tau)$ the demand flow. The state of the system is described by the following functions:

- $f_i(\tau)$, $i = a, b$—the input flow on arc i at time τ.
- $t_i(\tau)$, $i = a, b$—the travel time for drivers entering arc i at time τ.

Clearly, $f(\tau) = f_a(\tau) + f_b(\tau)$. We can assume also, without loosing too much in generality, that the travel time functions $t_i(\tau)$ are continuous in τ. Let us look first at the following dynamics: there exists a critical time $\hat{\tau}$ such that

$$f(\tau) \le \bar{f}_a \quad \text{for } \tau \le \hat{\tau}$$

and

$$\bar{f}_a < f(\tau) < \bar{f}_a + \bar{f}_b \quad \text{for } \tau > \hat{\tau}.$$

For simplicity, let us assume that $\int_{\hat{\tau}}^{\infty} (f(\tau) - \bar{f}_a)\, d\tau = \infty$. From Assumption 3, we know that before the critical time $\hat{\tau}$ there is no congestion:

$$f_a(\tau) = f(\tau), \quad f_b(\tau) = 0, \quad t_a(\tau) = \bar{t}_a, \quad t_b(\tau) = \bar{t}_b > \bar{t}_a, \quad \text{for } \tau \le \hat{\tau}.$$

What happens at $\tau = \hat{\tau}$? Since $t_i(\tau)$ are continuous, we will have $t_b(\tau) = \bar{t}_b > \bar{t}_a$ for some interval $[\hat{\tau}, \bar{\tau})$. During this period of time, the first arc will stay more attractive for the drivers, so,

$$f_a(\tau) = f(\tau), \quad \text{for } \tau \in [\hat{\tau}, \bar{\tau}).$$

However, during this period of time, the input flow of the first arc will be greater than the maximal output flow. This must result in a queue growing at the end of this arc. This queue will make the travel time $t_a(\tau)$ increasing. How long that can happen? Clearly, up to the moment the travel time on the first arc reaches the level of the travel time on the second arc:

$$t_a(\bar{\tau}) = \bar{t}_b.$$

Starting from time $\bar{\tau}$, the drivers will choose both arcs in such a way that the travel time on them remains the same. Therefore, for any $\tau > \bar{\tau}$ we have the following dynamic solution:

$$f_a(\tau) = \bar{f}_a, \quad f_b(\tau) = f(\tau) - \bar{f}_a, \quad t_a(\tau) = t_b(\tau) = \bar{t}_b, \quad \text{for } \tau > \bar{\tau}.$$

The above dynamics can be seen as a *transition process* between two stationary patterns in *travel time*. The particular dynamics of this process can be quite complicated, but it does not change the final stationary values of travel time. This state is stable since for any flow pattern $f(\tau)$ such that

$$\bar{f}_a < f(\tau) < \bar{f}_a + \bar{f}_b, \quad \text{for } \tau > \bar{\tau},$$

the equilibrium arc travel time, flow and queue on the first arc and the travel time on the second arc remain *constant over time*.

Let us look at the last variant of our solution, in which we have $f = \bar{f}_a$. In this case

$$f_a = \bar{f}_a, \quad f_b = 0,$$

but we cannot assign a specific value to the equilibrium travel time. We can only say that $t^e \in [\bar{t}_a, \bar{t}_b]$. This uncertainty has an obvious dynamic interpretation. If in our dynamic setting the integral $\int_{\hat{\tau}}^{\infty}(f(\tau) - \bar{f}_a)\,d\tau$ is finite and small enough, the travel time $t_a(\tau)$ can converge to some value from the interval $[\bar{t}_a, \bar{t}_b]$. But this value is not stable: any local change of the flow $f(\tau)$ will result in the change of the size of the queue and, consequently, in the stabilization of the travel time $t_a(\tau)$ at some new level.

Thus, let us make some intermediate conclusions.

- In our example we managed to find a stable stationary characteristics of dynamic assignment *without* any pre-defined travel time function. We only used the logical consequences of Assumptions 1, 2, 3.
- We have seen that the *arc performance model* is not necessarily a functional dependence. In our example it is represented as some bound on the arc travel time and the arc flow:

$$t_i \geq \bar{t}_i, \quad 0 \leq f_i(\tau) \leq \bar{f}_i, \quad i = a, b.$$

- Our equilibrium solution can be seen as stable stationary states with respect to *travel time*.

Note that, the hypothesis which characterizes the above stationary solutions allows a further characterization of the congestion pattern. Assume, for example, that we are in the intermediary regime $\bar{f}_a < f(\tau) < \bar{f}_a + \bar{f}_b$. In this case, the equilibrium travel time is $\bar{t}_b$ so that there is congestion on arc a. Denote $n_a(\tau)$ the occupancy of arc a. Since in the stationary regime the input and the output flows on the arc are the same, we conclude that the occupancy is a constant function of time:

$$n_a(\tau) = n_a.$$

Since the travel time on this arc is constant, from the FIFO principle it follows that

$$t_a = \frac{n_a}{\bar{f}_a}, \tag{1}$$

which could be understood as a kind of queuing model. If we assume a vertical queue, the number of running cars on arc a is $n_a^r = \bar{f}_a\bar{t}_a$. Therefore, the number of queuing cars is: $n_a^q = \bar{f}_a(t_a - \bar{t}_a)$. On arc b, there is no queuing cars.

Let us find the social optimum of our model (the optimum values are denoted by a superscript s). Assume that $f(\tau) \equiv f$. Note that at the social optimum, the travel time of each arc should be equal to the minimum travel time on this arc. Let us consider two cases, in which the stationary solution does exist.

If $f < \bar{f}_a$, all users are allocated to arc a, and the optimum average travel time is $t^s = \bar{t}_a$.

If $\bar{f}_a \leq f < \bar{f}_a + \bar{f}_b$, a flow $f_a^s = \bar{f}_a$ is allocated to arc a, and the remaining flow $f_b^s = f - \bar{f}_a$ is allocated to arc b. The optimum average travel time t^s, is:

$$t^s = \frac{\bar{t}_a\bar{f}_a}{f} + \frac{\bar{t}_b(f - \bar{f}_a)}{f} = \bar{t}_b - (\bar{t}_b - \bar{t}_a)\frac{\bar{f}_a}{f}.$$

Since in this case $t^e = \bar{t}_b$, then the relative saving Ω is:

$$\Omega \equiv \frac{t^e - t^s}{t^e} = \left(1 - \frac{\bar{t}_a}{\bar{t}_b}\right)\frac{\bar{f}_a}{f} > 0.$$

In this paper we will focus on the study of stationary equilibrium regimes (in travel time) for general transportation networks. We refer to these regimes as Stable Dynamics solutions of the traffic assignment problem. The analysis of traffic count data suggests that this approach is likely to be relevant as for peak and for off-peak hours. What will be ignored, is how the transportation system evolves from one stationary regime to another. Later on, we will prove that the existence of a stationary regime can be guaranteed for general networks under very mild assumptions and that similar conclusions could then be derived. However, before that we wish to develop our intuition with two more examples.

2.2. *Triangle network*

Let us consider a simple network with two origin nodes, ω_1 and ω_2, and one destination node δ. Node ω_1 is connected with δ by an arc with characteristics $\bar{t}_1$ and $\bar{f}_1$ and node ω_2 is connected with δ by the second arc with characteristics $\bar{t}_2$ and $\bar{f}_2$. Without loss of generality, we can assume that $\bar{t}_1 < \bar{t}_2$. Now, if the demand flows d_1 from ω_1 to δ and d_2 from ω_1 to δ satisfy the relation

$$d_1 < \bar{f}_1, \quad d_2 < \bar{f}_2,$$

then there is no congestion in the network. In this case

$$t_1^e = \bar{t}_1, \quad t_2^e = \bar{t}_2,$$

and the social cost in the network is as follows:

$$c^s = d_1\bar{t}_1 + d_2\bar{t}_2.$$

In what follows, we assume that

$$d_a + d_b > \bar{f}_1. \tag{2}$$

Let us modify our network. Namely, let us connect the nodes ω_1 and ω_2 by a new very short and efficient arc with the characteristics $\bar{t}_3$ and $\bar{f}_3$. We assume that these characteristics satisfy the following relations:

$$\bar{t}_3 + \bar{t}_1 < \bar{t}_2, \quad \bar{f}_3 > \bar{f}_1.$$

What is the impact of this modification? Clearly, it changes nothing for drivers of OD-pair (ω_1, δ). However, for the drivers of OD-pair (ω_2, δ) it creates a new shortest path. Therefore these drivers will try to use it. But, since the capacity of the first arc is not enough to carry out the total demand flow (2), the drivers from ω_2 will create a queue on the first arc. The size of this queue will be growing up to the moment the travel time along the route $\omega_2 \to \omega_1 \to \delta$ becomes equal to $\bar{t}_2$. Since there is no congestion on the new arc, the travel time on the first arc must be as follows:

$$t_1^e = \bar{t}_2 - \bar{t}_3.$$

Thus, the above modification of the network results in the following equilibrium travel times and arc flows:

$$
\begin{aligned}
&t_1^e = \bar{t}_2 - \bar{t}_3, \quad f_1 = \bar{f}_1, \\
&t_2^e = \bar{t}_2, \quad f_2^e = d_1 + d_2 - \bar{f}_2, \\
&t_3^e = \bar{t}_3, \quad f_3^e = \bar{f}_1 - d_1.
\end{aligned}
$$

The equilibrium travel time for the drivers of OD-pair (ω_2, δ) is not improved. But the travel time for traveling from ω_1 to δ becomes worse. The new social cost is as follows:

$$c_1^s = d_1 \cdot (\bar{t}_2 - \bar{t}_3) + d_2 \cdot \bar{t}_2 > c^s.$$

Thus, nobody is better off after the above modification and some drivers are worse off. Such a phenomenon is known in transportation science as *Braess paradox* (see, e.g. Steinberg and Stone, 1988). However, the classical form of that paradox corresponds to the static model with linear arc performance functions (we give a short overview of these models in Section 3). Moreover, it is derived for a very special network (we consider this network in Section 2.3). Up to now the theoretical conditions for arising such negative effects in general static models are still not well completely understood. The example we have seen in this section seems to be new. Moreover, it corresponds to the simplest network structure and it is much easier for complete theoretical analysis. An interesting question is why this triangular network was never considered as an example for the Braess paradox. The answer is quite intriguing: It appears that for the triangular network the negative impact *cannot* be observed in a static model with strongly increasing linear travel time functions.

Note that the above equilibrium solution can be supported by the same type of dynamic analysis of the stable stationary states as it was done in Section 2.1.

2.3. Braess paradox revisited

Let consider a network with four nodes O, B_1, B_2 and D. Node O is an origin and node D is a destination. We first consider the following network: node O and B_1 are connected by arc a_1, and B_1 and D are connected by arc a_2. Similarly, node O and B_2 are connected by arc a_3, and B_2 and D are connected by arc a_4. Let f denote the demand flow, i.e. the flow going from the origin to the destination. Since we are interested in a stationary solution, we treat f as a constant function of time.

In order to simplify the discussion below, we assume that there is never congestion on arcs a_2 and a_3:

$$\bar{f}_2 \gg f, \quad \text{and} \quad \bar{f}_3 \gg f.$$

Therefore, there are two routes from the origin node O to the destination node D. Route 1 defined by $\{O \to B_1 \to D\}$ has a minimum travel time and a capacity given by

$$\bar{t}_{12} = \bar{t}_1 + \bar{t}_2, \quad \text{and} \quad \bar{f}_{12} = \bar{f}_1$$

while route 2 defined by $\{O \to B_2 \to D\}$ has a minimum travel time and a capacity given by:

$$\bar{t}_{34} = \bar{t}_3 + \bar{t}_4, \quad \text{and} \quad \bar{f}_{34} = \bar{f}_4.$$

We assume that $\bar{t}_{12} > \bar{t}_{34}$.

We wish to solve for a stable dynamic equilibrium. Such an equilibrium exists if $f < \bar{f}_1 + \bar{f}_4$, a condition that we assumed in the subsequent analysis. There are two cases to consider.

If $f < \bar{f}_4$, there is no congestion on the network and all divers use the shortest path (route 2), and the equilibrium travel time is $t_{\mathrm{I}}^e = \bar{t}_{34}$.

If $f \in (\bar{f}_4, \bar{f}_1 + \bar{f}_4)$, then congestion occur on route 2 (the shortest), and arc a_4 is congested. The equilibrium travel time is: $t_{\mathrm{I}}^e = \bar{t}_{12}$.

We now consider the impact of an additional arc, denoted arc a_5, with $\bar{f}_5 \gg f$, connecting node B_1 and B_2. There is now a third route from O to D: $\{O \to B_1 \to B_2 \to D\}$. We further assumed that:

$$\begin{cases} \bar{t}_1 < \bar{t}_3; \quad \bar{t}_2 > \bar{t}_4 \\ \bar{t}_5 < \min\{\bar{t}_3 - \bar{t}_1, \bar{t}_2 - \bar{t}_4\} \end{cases} \tag{3}$$

We restrict the analysis to the case where $f \in (\max(\bar{f}_1, \bar{f}_4), \bar{f}_1 + \bar{f}_4)$; in this case, all three routes will be used. Condition (3) implies that arc a_1 is congested with $t_1^e = \bar{t}_3 - \bar{t}_5$ (since both routes $\{O \to B_1 \to B_2\}$ and $\{O \to B_2\}$ are used). Similarly, arc a_4 is congested with $t_4^e = \bar{t}_2 - \bar{t}_5$. The equilibrium travel time, the same on the three routes, is:

$$t_{\mathrm{II}}^e = \bar{t}_3 + \bar{t}_2 - \bar{t}_5. \tag{4}$$

Since arcs a_1 and a_4 are congested, the equilibrium flow satisfy: $f_1^e = \bar{f}_1$ and $f_4^e = \bar{f}_4$.

Without the additional arc, the equilibrium travel time is $t_{\mathrm{I}}^e = \bar{t}_{34}$. With the additional link, the equilibrium travel time is $t_{\mathrm{II}}^e = \bar{t}_3 + \bar{t}_2 - \bar{t}_5$. Using condition (3), it is clear that $t_{\mathrm{I}}^e < t_{\mathrm{II}}^e$. Therefore, the additional arc, a_5, increases the equilibrium travel time. The faster is the additional arc, a_5, the more severe is the increase in the equilibrium travel time. The worse case occurs when the additional link has an infinite speed. In this case, nodes B_1 and B_2 coincide.

When $\bar{t}_5 = 0$, the equilibrium travel time is $t_{\mathrm{II}}^e = \max\{\bar{t}_1, \bar{t}_3\} + \max\{\bar{t}_2, \bar{t}_4\}$ ($t_{\mathrm{II}}^e = \bar{t}_3 + \bar{t}_2$, if condition (3) holds). Interestingly, the equilibrium travel time is increased when the connectivity of the network increases. This is because the equilibrium condition adjust the travel time to the largest travel time, used as a benchmark. Such an effect is also in operation in the standard presentation of Braess paradox, although it appears somewhat hidden by the algebraic derivations. Here we were able to derive this paradox using only logical arguments.

In the next sections, we show how to extend the Stable Dynamic formulation on the general networks. However, first of all we need to introduce the network notation. We will give also a brief description of the Static network assignment models, which pretend to compute the stationary equilibrium solutions.

3. Static network assignment models: Main results and Beckmann approach

3.1. Main results in static models

Let us introduce some notation. We work with a network $\mathcal{R}$ comprised by the set of nodes $\mathcal{N}$ and the set of directed arcs $\mathcal{A}$:

$$\mathcal{R} = \{\mathcal{N}, \mathcal{A}\}, \quad \mathcal{N} = \{1, \dots, n\}, \quad \mathcal{A} = \{1, \dots, m\}.$$

To each $\alpha \in \mathcal{A}$ we relate a pair of nodes $i_\alpha, j_\alpha \in \mathcal{N}$. For each arc $\alpha \in \mathcal{A}$ denote $f^{(\alpha)}$ the total flow of drivers travelling along this arc. It is convenient to treat the set of all arc flows as a column vector:

$$f = \left(f^{(1)}, \dots, f^{(m)}\right)^T \in R^m.$$

Again, we will try to treat f as a constant function of time.

In order to describe the travel cost of a trip in the network, for each arc α we introduce the *cost function*:

$$c^{(\alpha)}(f), \quad f \in D \subseteq R^m, \quad \alpha = 1, \dots, m,$$

where D is a natural open convex domain of those functions. It is convenient to treat the set of all cost function as a vector function:

$$c(f) = \left(c^{(1)}(f), \dots, c^{(m)}(f)\right)^T.$$

In the simplest case, each component of this vector function depends only on the flow at the corresponding arc. However, we will see that under some natural assumptions on the function $c(f)$ we can treat also the interaction of the flows at different arcs.

The presence of the cost functions in the traffic assignment model represents the main difference between the standard static models and the Stable Dynamics approach. In Section 3.3 we will see that these objects create a lot of troubles for any kind of dynamic interpretation of the results. However, first of all we need to see the story up to the end.

For network $\mathcal{R}$ we define the set of origin-destination pairs (O-D pairs):

$$\mathcal{OD} = \{\pi_k = (\omega_k, \delta_k) : \omega_k, \delta_k \in \mathcal{N},\ \omega_k \neq \delta_k,\ k = 1, \dots, p\}.$$

Each O-D pair π_k generates a demand d_k. This demand is traditionally considered as an average flow of drivers, which need to travel from node ω_k to another node δ_k; so the demand is a non-negative real number.

Further, for each O-D pair π_k we can introduce the *finite* set of all possible routes connecting ω_k with δ_k:

$$\mathcal{R}_k = \{a_{k,r} \in R^m, r = 1, \dots, r_k\},$$

where the component $a_{k,r}^{(\alpha)} = 1$ if the arc α is included in the route r; otherwise this component is zero. For the sake of simplicity we denote A_k the matrix composed by all column vectors $a_{k,r}$:

$$A_k = \left(a_{k,1}, \dots, a_{k,r_k}\right).$$

Finally, for each O-D pair π_k we introduce the set of feasible *route flow partitions*:

$$\Delta_k = \left\{ F_k \in R_+^{r_k} : \sum_{r=1}^{r_k} F_k^{(r)} = d_k \right\}.$$

The component $F_k^{(r)}$ of the vector $F_k \in \Delta_k$ tells us which flow goes from the origin ω_k to the destination δ_k along the route r from the corresponding set of routes R_k.

The above notation allows us to describe the loading of the network $\mathcal{R}$ in a convenient form. Indeed, if for O-D pair π_k we choose some route flow partition $F_k \in \Delta_k$, then the total impact of these flows in the arc flow pattern in the network is as follows:

$$f_k = \sum_{r=1}^{r_k} F_k^{(r)} a_{k,r} = A_k F_k .$$

Denote $F = (F_1^T, \ldots, F_p^T)^T$ and $\Delta = \prod_{k=1}^{p} \Delta_k$. Then the total arc flow pattern in the network is

$$f = \sum_{k=1}^{p} A_k F_k \equiv AF,$$

where $A = (A_1, \ldots, A_p)$.

Now we can define the travel cost in the network as a function of the full flow partition vector F. Indeed, if we choose some $F \in \Delta$, then the travel cost of arc α is

$$c^{(\alpha)}(f) = c^{(\alpha)}(AF).$$

Then, for O-D pair π_k the cost of travelling along some route $a_{k,r} \in \mathcal{R}_k$ is as follows:

$$C_k^{(r)}(F) = \sum_{\alpha=1}^{m} c^{(\alpha)}(AF) a_{k,r}^{(\alpha)} \equiv \langle c(AF), a_{k,r} \rangle.$$

Thus, the route cost vector for this O-D pair is defined as

$$C_k(F) = A_k^T c(AF).$$

Finally, the total route cost vector for all O-D pairs $C(F) = (C_1^T(F), \ldots, C_p^T(F))^T$ can be represented as

$$C(F) = A^T c(AF).$$

Denote $\Delta_D = \{F \in \Delta : AF \in D\}$. Now we can write down the standard *equilibrium traffic assignment* problem:

$$\begin{gathered}\textit{Find a flow partition vector } F^* \in \Delta_D \textit{ such that}\\ \textit{for any O-D pair } \pi_k \textit{ we have } (F_k^*)^{(l)} > 0 \Rightarrow C_k^{(l)}(F^*) = \min_{1 \le r \le r_k} C_k^{(r)}(F^*).\end{gathered} \tag{5}$$

It is well known (see Nagurney, 1993a) that the equilibrium problem (5) can be written in the form of *variational inequality*. For completeness of the paper we provide this statement with a simple proof.

Theorem 1. *The problem* (5) *is equivalent to finding a strong solution F^* the variational inequality* (6), *that is*

$$\text{Find } F^* \in \Delta_D : \langle C(F^*), F - F^* \rangle \geqq 0 \quad \text{for all } F \in \Delta_D. \tag{6}$$

Proof: Indeed, let F^* be a solution to variational inequality (6). Consider an arbitrary O-D pair π_k. Assume that $(F_k^*)^{(l)} > 0$ and for some other route $r \in \mathcal{R}_k$ we have $C_k^{(r)}(F^*) < C_k^{(l)}(F^*)$. Let us choose $\varepsilon > 0$ small enough, such that all coordinates of the vector

$$\hat{F}_k = \left((F_k^*)^{(1)}, \ldots, (F_k^*)^{(l)} - \varepsilon, \ldots, (F_k^*)^{(r)} + \varepsilon, \ldots, (F_k^*)^{(r_k)}\right)^T$$

are non-negative and $\hat{F}_k \in \Delta_k$. Therefore, for ε small enough

$$\hat{F} = ((F_1^*)^T, \ldots, (\hat{F}_k)^T, \ldots, (F_p^*)^T)^T \in \Delta_D.$$

At the same time,

$$\langle C(F^*), \hat{F} - F^* \rangle = \langle C_k(F^*), \hat{F}_k - F_k^* \rangle = C_k^{(l)}(F^*) \cdot (-\varepsilon) + C_k^{(r)}(F^*) \cdot \varepsilon < 0.$$

This contradicts to our assumption that F^* is a solution to (6).

Vice versa, let F^* be a solution to (5). Denote

$$u_k = \min_{1 \le r \le r_k} C_k^{(r)}(F^*), \quad k = 1, \ldots, p.$$

Since any vector $F \in \Delta_D$ has non-negative components, we obtain:

$$\begin{aligned}
\langle C(F^*), F \rangle &= \sum_{k=1}^{p} \langle C_k(F^*), F_k \rangle = \sum_{k=1}^{p} \sum_{r=1}^{r_k} C_k^{(r)}(F^*) F_k^{(r)} \\
&\geqq \sum_{k=1}^{p} \left(u_k \sum_{r=1}^{r_k} F_k^{(r)} \right) = \sum_{k=1}^{p} u_k d_k \\
&= \sum_{k=1}^{p} \left(u_k \sum_{r=1}^{r_k} (F_k^*)^{(r)} \right) = \sum_{k=1}^{p} \sum_{r=1}^{r_k} C_k^{(r)}(F^*)(F_k^*)^{(r)} \\
&= \sum_{k=1}^{p} \langle C_k(F^*), F_k^* \rangle = \langle C(F^*), F^* \rangle. \qquad \square
\end{aligned}$$

Thus, the equilibrium traffic assignment problem (5) is completely equivalent to the variational inequality problem (6). In order to ensure the existence of the solution to the latter problem, we need some assumptions.

Definition 1. The travel cost vector function $c(x)$ is called monotone if

$$\langle c(x) - c(y), x - y \rangle \geqq 0 \quad \forall x, y \in D.$$

If this inequality is strict, then the function is called strictly monotone. Function $c(f)$ is called closed if for any $y \in D$ we have $\langle c(x), x - y \rangle \to +\infty$ as $x \to \partial D$.

Note that the closed monotone vector functions may be discontinuous and unbounded on their domain. Nevertheless, this property ensures some general existence results for corresponding variational inequalities. In the following theorem we use some results from the theory of monotone variational inequalities (see Nesterov and de Palma, 2000).

Theorem 2. *Let the travel cost function $c(x)$ be closed and monotone. If $\Delta_D \neq \emptyset$, then there exists a weak solution F_* to the variational inequality problem* (6), *that is*

$$F_* \in \Delta_D : \langle C(F), F - F_* \rangle \geqq 0 \quad \forall F \in \Delta_D. \tag{7}$$

If $c(x)$ is continuous, then F_* is also a strong solution to (6). *If $c(x)$ is strongly monotone, then in both cases the solution is unique.*

Proof: Since $C(F) = A^T c(AF)$, the statement of the theorem follows from the standard results (see, e.g. Nesterov and de Palma, 2000). □

In the next section we will see some important examples of the closed monotone travel cost functions, which are discontinuous and unbounded on their domain.

3.2. Beckmann model

To conclude this section, let us discuss a particular case of the general equilibrium traffic assignment model (5), which is called *Beckmann model* (see Beckmann et al., 1956). In this model it is assumed that for each arc the travel cost function $c^{(\alpha)}(f)$ only depends on the flow on this arc:

$$c^{(\alpha)}(f) \equiv c^{(\alpha)}\big(f^{(\alpha)}\big), \quad \alpha = 1, \ldots, m. \tag{8}$$

Let us assume also that $0 \in D$. Under these assumptions the variational inequality problem (6) can be written in much simpler form. Indeed, let us define the following functions:

$$\sigma^{(\alpha)}(u) = \int_0^u c^{(\alpha)}(\tau)\, d\tau, \quad \alpha = 1, \ldots, m.$$

Note that $(\sigma^{(\alpha)}(u))' = c^{(\alpha)}(u)$. Therefore, in view of Assumption 1, each component $\sigma^{(\alpha)}(u)$ is a *convex* function in u. Hence, the function

$$\sigma(f) = \sum_{\alpha=1}^{m} \sigma^{(\alpha)}\big(f^{(\alpha)}\big)$$

is convex in f. Let us look now at the function

$$S(F) = \sigma(AF), \quad \operatorname{dom} S = \{F : AF \in D\}.$$

This function is closed and convex. Let us assume for a moment that $c(f)$ is continuous. Then $S(F)$ is differentiable and

$$\nabla S(F) = A^T c(AF) \equiv C(F).$$

Thus, the variational inequality condition (6) can be written as

$$\langle \nabla S(F^*), F - F^* \rangle \geqq 0 \quad \forall F \in \Delta_D.$$

And that is exactly the optimality condition for the following convex minimization problem:

$$S^* = \min_F \{S(F) : F \in \Delta_D\} = \min_{f,F} \{\sigma(f) : f = AF,\ F \in \Delta_D\}. \tag{9}$$

Thus, we come to the following statement.

Theorem 3. *Let the travel cost function $c(f)$ be closed and monotone. If it satisfies the condition* (8), *then any solution to the convex optimization problem* (9) *is a weak solution of the variational inequality* (7). *If $c(f)$ is continuous, then any solution of this problem is an equilibrium flow partition for the problem* (5).

Note that the optimization formulation (9) has at least two advantages with respect to the variational inequality problem (6). First of all, in general the optimization problems are much easier from the computational point of view than the variational inequalities (see Nemirovsky and Yudin, 1983). The second advantage is related to the number of variables in (9) and (6). Both of these problems are posed with respect to the total flow partition vector F. This means, that in order to form the matrix A in these problems, we need to enumerate all possible paths in the network. Even for relatively small networks the number of the paths is exponentially large. Therefore both problems are solvable numerically only for a very small network. Nevertheless, since the problem (9) is an *optimization* problem, we can rewrite it in an equivalent *dual* form with a reasonable number of variables. We will see that this dual problem is solvable by the numerical schemes and that the primal variables F (or a part of this vector) can be also computed.

Let us derive now the dual form of the problem (9). First of all, we need to introduce the concept of *conjugate* function.

Definition 2. For convex function $\varphi(x) : \mathrm{dom}\, \varphi \to R$, the function

$$\varphi_*(s) = \sup\{\langle s, x\rangle - \varphi(x) : x \in \mathrm{dom}\, \varphi\}$$

is called conjugate to φ.

Note that the function $\varphi_*(s)$ is always convex and under very mild assumptions we have $(\varphi_*)_* = \varphi$. This means that

$$\varphi(x) = \sup\{\langle s, x\rangle - \varphi_*(s) : s \in \mathrm{dom}\, \varphi_*\}.$$

For our goals we need to work only with conjugate functions of one variable. Namely, we need to define

$$\sigma_*^{(\alpha)}(\tau) = \sup\left\{\tau u - \sigma^{(\alpha)}(u) : u \in \mathrm{dom}\, \sigma^{(\alpha)}\right\}, \quad \alpha = 1, \ldots, m. \tag{10}$$

In the next section we will consider some typical examples of the functions $\sigma^{(\alpha)}(\cdot)$, and we will see that in many cases the corresponding conjugate functions can be computes in a closed form.

We need to introduce also the *shortest path functions* in the network $\mathcal{R}$. Let us fix some travel time on the arcs of the network:

$$t = \left(t^{(1)}, \ldots, t^{(m)}\right)^T \in R^m.$$

Denote $T_{(i,j)}(t)$ the shortest path distance between the nodes i and j with respect to the arc travel time pattern t. For an O-D pair $\pi_k = (\omega_k, \delta_k)$ denote $T_k(t) = T_{(\omega_k,\delta_k)}(t)$. It is easy to see that

$$T_k(t) = \min_{1 \leqq r \leqq r_k} \langle a_{k,r}, t \rangle,$$

so it is a *concave* piece-wise linear function of t.

Using the functions (10) and the travel time functions $T_k(t)$, we can define the problem dual to (9):

$$S_* = \max \left\{ \sum_{k=1}^{p} d_k T_k(t) - \sum_{\alpha=1}^{m} \sigma_*^{(\alpha)}\big(t^{(\alpha)}\big) : t \in \mathcal{F} \right\}, \tag{11}$$

where $\mathcal{F} = \prod_{\alpha=1}^{m} (\text{dom } \sigma_*^{(\alpha)})$.

Theorem 4. *If $\Delta_D \neq \emptyset$, then the optimal solutions F^* and t^* for the problems* (9) *and* (11) *exist and $S^* = S_*$. Moreover, t^* is the vector of optimal dual multipliers for the equality constraints $f = AF$ in* (9).

Proof: Indeed, for the Beckmann model we have:

$$\begin{aligned}
S^* &= \min_{f,F} \left\{ \sum_{\alpha=1}^{m} \sigma^{(\alpha)}\big(f^{(\alpha)}\big) : f = AF,\, F \in \Delta \right\} \\
&= \min_{f,F} \left\{ \sum_{\alpha=1}^{m} \max_{t^{(\alpha)} \in \text{ dom } \sigma_*^{(\alpha)}} \left[f^{(\alpha)} t^{(\alpha)} - \sigma_*^{(\alpha)}\big(t^{(\alpha)}\big)\right] : f = AF,\, F \in \Delta \right\} \\
&= \max_{t \in \mathcal{F}} \min_{f,F} \left\{ \langle f, t \rangle - \sum_{\alpha=1}^{m} \sigma_*^{(\alpha)}\big(t^{(\alpha)}\big) : f = AF,\, F \in \Delta \right\} \\
&= \max_{t \in \mathcal{F}} \left[\min_{f,F} \{ \langle f, t \rangle : f = AF,\, F \in \Delta \} - \sum_{\alpha=1}^{m} \sigma_*^{(\alpha)}\big(t^{(\alpha)}\big) \right].
\end{aligned}$$

It remains to note that

$$\langle f, t \rangle = \langle AF, t \rangle = \langle F, A^T t \rangle = \sum_{k=1}^{p} \big\langle F_k, A_k^T t \big\rangle.$$

Hence,

$$\min_{f,F} \{ \langle f, t \rangle : f = AF,\, F \in \Delta \} = \sum_{k=1}^{p} \min\big\{ \big\langle F_k, A_k^T t \big\rangle : F_k \in \Delta_k \big\} = \sum_{k=1}^{p} d_k T_k(t). \qquad \square$$

Thus, instead of solving the problem (9) of very high dimension, we can solve a non-smooth convex optimization problem (11), where the number of variables is equal to the number of arcs in the network. The objective function of the latter problem includes the shortest path functions, which can be computed very efficiently (see, for example, Knuth, 1979).

3.3. Scope and limitation of Beckmann model

Traditionally, the Beckmann model, described in the previous section, is used for the analysis of congestion arising in *private* transportation (flows of private cars in urban or interurban road structures). In this case, we have to interpret the arc travel cost function $c^{(\alpha)}(f^{(\alpha)})$ as *travel time* spent by the drivers on arc α. There were proposed many different forms of such travel time functions. All of them lead to a problem which can be solved numerically. However, the question is how much such models are consistent with the physical laws regulating traffic congestion.

Indeed, let us look at the main assumption in the Beckmann model. It reads as follows:

The travel time on each arc is a *non-decreasing* function of flow.

Let us try to examine this statement in more details, by discussing four different types of arguments.

First, if we agree that the static models should provide us with a stationary regime of a dynamic process, then necessarily the flow and the density (or occupancy) of drivers on a particular arc, as well as the travel time on this arc are constant over time. Note that at a stationary regime, the following fundamental identity necessarily holds:

$$\textit{flow} = \textit{speed} \times \textit{density}.$$

Let us assume for a moment that the density of drivers is constant over time. We examine below the impact of an exogenous change in the (arrival) flow. Since the density is constant, an *increase* of the flow results in a proportional increase of the speed and, consequently, in a *decrease* of the travel time, a contradiction. Thus, in order to justify this main assumption (see above), we need to assume also that the density is increasing much faster than the flow. Unfortunately, this is not always possible since the feasible range of variation of the density on the roads is not too large. The density is bounded from above by $\rho_{\max} = 1000\,\mathrm{m}/(6\,\mathrm{m/veh}) \approx 165$ veh/km, where 6 m is the average length occupied by a car in a stop and go movement. The corresponding speed is about 3 kilometers an hour. On the other hand, for density of about 33 veh/km, which is five-six times smaller than $\rho_{\max}$, we can already observe free flow speed on urban freeways (about 90 km/hr). This range of variation of the density (from 1 to 6) clearly does not allow us to model in a consistent way the significant drop of the speed (by a factor of 30) which parallels the increase in the flow.

Second, applied static models allow flows to be larger than the capacity. Then necessarily the additional flow has to be assigned to the next time period, so that the regime in this case is not stationary. In the next section, we will discuss another approach, Stable Dynamics, in which we only consider *stationary regimes*. This is possible by the fact that we assume the capacity constraint could never be violated.

Third, the basic observation of traffic count suggests that a small flow can correspond to two opposite situations. Either the road is not attractive and the drivers do not use it; then we observe the free traffic conditions on this arc (this is known as the stable branch of the fundamental diagram). Or, the road is heavily congested and the speed is very small (and as a consequence the flow is very small). Thus, we cannot say that the (input) flow is the *only variable* which is responsible for the travel time on the arc. Traffic count data widely suggest that the same flow may support more than one travel time pattern (unstable branch). In other words, the relation between travel time and flow is not a priori a function but a *correspondence*. Of course, a dynamic model could easily handle the fact that the same flow may be consistent with more than one speed regime. As shown in the next section, the stable dynamic approach is also consistent with those basic traffic count observations.

Four, the second assumption (8) in Beckmann's model tells us that only the flow on a particular arc is responsible for the travel time at the same arc. We have seen above that this assumption is not necessarily satisfied, since the relation between flow and travel time is not one to one. This should not be a surprise for the reason described below. Indeed, the travel condition of an arc is related to the travel conditions of this arc with respect to traffic conditions on the surrounding road network, that is on the nearest alternative roads. This does not say that because of road intersections the flow on an arc may slow down the cars of the intersecting roads. This is true, but our suggestion is different. We wish to say that the equilibrium travel conditions on an arc depend, a priori on the *equilibrium* travel conditions *on all substitue roads*. This implies that any relation between travel time and flow, defined on a specific arc, is incomplete. Indeed the travel time is an outcome of the equilibrium conditions and it could not be understood out of the equilibrium context, but only along the equilibrium paths.

The above discussion shows that, despite its theoretical attraction, Beckmann's model is not consistent with a stationary regime and could not explain that the travel time is not a function of the flow. Thus, we come to the following natural question:

Is there any transportation network model
for which Beckmann's assumptions are rigourosly satisfied?

In fact the authors came to a conclusion that the Beckmann assumptions are very much suitable for a rigorous analysis of *public* transportation with traffic congestion. However, since this topic is quite far from the main scope of this paper, we put our arguments in Appendix 1. In the next Section 4, we present an alternative approach for finding the stationary equilibrium solutions for traffic congestion, which is free from the above objections.

4. Stable dynamics: Derivation for a network

4.1. Structure of equilibrium flow

Consider now a general transportation network. Let assume that the arc travel time pattern $t = \{t_\alpha\}_{\alpha\in\mathcal{A}}$ is given. Then, for each O-D pair π_k, we can compute the shortest-path travel time function $T_k(t)$, with

$$T_k(t) = \min\{\langle a_{k,r}, t\rangle, r = 1, \ldots, r_k\}.$$

Therefore, $T_k(t)$ is a concave piece-wise linear function of t which is defined for any $t \in R^m$.

Recall that for any function $f(x)$, which is concave on R^m, at each point a superdifferential, denoted by $\partial f(x)$ can be defined. This is a closed convex set such that for any

$g \in \partial f(x)$, the following inequality holds:

$$f(x) \leq f(x) + \langle g, y - x \rangle, \quad \forall y \in R^m.$$

We now look at the superdifferential of $T_k(t)$. Define $I_k(t)$ as

$$I_k(t) = \{r \in [1 \ldots r_k] : \langle a_{k,r}, t \rangle = T_k(t)\}.$$

In other words, $I_k(t)$ is the set of all shortest paths with respect to t, which connect the origin node ω_k to the destination node δ_k. Then

$$\partial T_k(t) = \text{Conv}\{a_{k,r}, \ r \in I_k(t)\}.$$

This set allows us to characterize in a very compact form the equilibrium flows induced by the O-D pair π_k in the network $\mathcal{R}$.

Lemma 5. *The flow vector f_k is compatible with Assumption* 1 *if and only if there exists some $g \in \partial T_k(t)$ such that*

$$f_k = d_k g.$$

In the sequel, we call such a vector f_k the equilibrium flow of the O-D pair π_k. Note that the cumulative arc flow f is just a summation of all OD-flows:

$$f = \sum_{k \in \mathcal{OD}} f_k.$$

We call f the equilibrium flow if it can be represented as a sum of equilibrium flows of all O-D pairs. Note that the equilibrium flows are defined with *respect to* the arc travel time pattern t.

Consider now the cost function

$$C(t) = \sum_{k \in \mathcal{OD}} d_k T_k(t).$$

Theorem 6. *The arc flow vector $f \in R^m$ is an equilibrium flow with respect to the arc travel time pattern t if and only if*

$$f \in \partial C(t). \tag{12}$$

This theorem allows us to answer some interesting questions. Let the demand flow $\{d_k\}_{k \in \mathcal{OD}}$ be known.

1. Given the arc travel time vector t, we can describe all possible equilibrium flows, which can arise in the network. That is

$$f_k \in d_k \partial T_k(t), \quad k \in \mathcal{OD},$$
$$f = \sum_{k \in \mathcal{OD}} f_k.$$

2. Given the arc flow vector f, we can check whether it is possible to find an arc travel time vector t, with respect to which f is an equilibrium flow.

Lemma 7. *The arc flow f is an equilibrium flow in the network $\mathcal{R}$ if and only if the following optimization problem*

$$\max_t [C(t) - \langle f, t\rangle] \tag{13}$$

admits a non-negative solution t^e.

This statement shows that the information from the traffic counters can help to reconstruct the equilibrium travel time in the network. If all arc flows are known, this data is enough in order to find the equilibrium travel time. However, even a partial knowledge can help (see Section 4.4).

Note that the results presented in this section are based *only* on Assumptions 1, 2 and 3. In particular, the description of arc performance is only based on parameters which are directly measurable. Interestingly, we never use an hypothesis on arc performance, when arcs are congested.

4.2. Max-flow model

Let us show how we can get the equilibrium solutions discussed in Section (4.1). We use the following max-flow performance model:

$$t_\alpha \geq \bar{t}_\alpha, \quad 0 \leq f_\alpha \leq \bar{f}_\alpha, \quad \alpha \in \mathcal{A}. \tag{14}$$

Theorem 8. *The arc travel time t^e and the arc flow vector f^e is an equilibrium solution of the model* (14) *f and only if there is a solution to the problem*

$$\max_t [C(t) - \langle f, t\rangle : t \geq \bar{t}], \tag{15}$$

and $f^e = \bar{f} - s^e$, where s^e is a vector of optimal dual multipliers for the inequality constraints in (15).

4.3. Dual max-flow model

Note that (15) is a convex optimization problem. Therefore, as discussed below, it can be reformulated in an equivalent dual form.

In order to perform this task, it is convenient to introduce for each origin $\omega_k \in \mathcal{O}$ a demand flow vector $d_{\omega_k} \in R^n$, where $n = |\mathcal{N}|$. Each component of this vector is the demand flow from ω_k to the corresponding node. Denote by $E \in R^{nxm}$ the incidence matrix of the network $\mathcal{R}$:

$$E_{\omega_k \alpha} = \begin{cases} 1 & \text{if arc } \alpha \text{enters into node } \omega_k \\ -1 & \text{if arc } \alpha \text{ goes out of node } \omega_k, \omega_k = 1, \dots, n, \quad \alpha = 1, \dots, m. \\ 0 & \text{otherwise.} \end{cases}$$

The problem dual to (15) can be written in terms of arc flow vectors $f_i \in R^m$, generated by the origins $\omega_k \in \mathcal{O}$.

Lemma 9. *The problem*

$$\begin{aligned} \min_{f, f_\alpha} & \langle f - \bar{f}, \bar{t} \rangle \\ \text{s.t.} \quad & f = \sum_{\omega_k \in \mathcal{O}} f_{\omega_k} \leq \bar{f}, \\ & E f_{\omega_k} = d_{\omega_k}, \quad \omega_k \in \mathcal{O}, \\ & f_{\omega_k} \geq 0, \quad \omega_k \in \mathcal{O}, \end{aligned} \tag{16}$$

is dual to (15).

Thus the problem dual to (15) is the *minimal cost multi-commodity transportation problem* with bounded arc capacities. To the best of our knowledge, this problem was never considered for finding a user equilibrium in transportation systems. Traditionally, problems of that type are used for finding a system optimum. However, note that in our framework the solution of this problem gives us only the flow pattern in the congested network. Intuitively it seems reasonable that, when congestion occurs, the drivers increase the set of used paths in a monotone way, starting from the free traffic shortest path. The equilibrium arc travel times arise in this problem as the optimal dual multipliers from the inequalities $f \leq \bar{f}$. Note that the problem (16) belongs to the class of models studied by Larson and Patrikson (1997) (see, also Patrickson (1994) and Yang and Lam (1996)). However, in our approach we derive the model directly from the behavioral assumptions. Therefore the interpretation of the equilibrium solutions becomes much easier.

It is interesting that the problem (16) provides us also with a social optimum solution. Indeed, if f^e is a solution of this problem, then $\langle f^e, \bar{t} \rangle$ is the optimal social cost. Thus, the equilibrium solution differs from the social optimum solution only by the queue created by the drivers along the most attractive routes (see the discussion of two routes in parallel, for an illustration). These queues are clearly inefficient from the social point of view and they are null at the social optimum. However, they increase the travel time along the best routes up to the equilibrium level. As we have seen, the flow pattern for the user equilibrium and the social optimum is the same.

4.4. Mixed max-flow model

At the end of Section 4.1, it was mentioned that the complete knowledge of the arc flows in the network allow to recover the equilibrium travel time and the equilibrium OD-flows without any additional information for the arc performance, once they are congested. However, in real networks, the measurements of traffic counters are typically only available for a small number of arcs. In this case, we can combine the available information with the max-flow arc performance model.

Let consider the following model:

$$\begin{aligned} & f_\alpha = \hat{f}_\alpha & & \alpha \in \mathcal{C} \\ & 0 \leq f_\alpha \leq \bar{f}_\alpha, t_\alpha \geq \bar{t}_\alpha & & \alpha \in \mathcal{A} \setminus \mathcal{C}. \end{aligned} \tag{17}$$

In the above model, the set $\mathcal{C}$ corresponds to a subset of arcs, for which traffic count information does exist.

Theorem 10. *The arc travel time t^e and the arc flow f^e is an equilibrium solution to the model* (17) *if and only if t^e is a non-negative solution to the problem*

$$\max_t \left[C(t) - \sum_{\alpha \in \mathcal{C}} \hat{f}_\alpha t_\alpha - \sum_{\alpha \in \mathcal{A} \setminus \mathcal{C}} \bar{f}_\alpha t_\alpha : t_\alpha \geq \bar{t}_\alpha, \alpha \in \mathcal{A} \setminus \mathcal{C} \right], \tag{18}$$

and $f^e_\alpha = \bar{f}_\alpha - s^e_\alpha$, $\alpha \in \mathcal{A} \setminus \mathcal{C}$, where s^e_α are the optimal dual multipliers for the inequality constraints in (18).

Note that the problem (18) may be unsolvable. In this case the observed traffic counts contradict our choice of demand flows.

5. Concluding remarks

In Beckmann's type models, it is possible to consider situations with unbounded travel time functions. In our models, the equilibrium flows on each arc never exceeds the limiting capacity of this arc. When the flow is stabilized at the capacity level, the queue on this arc starts to grow. The level on which the queue is adjusted does not depend on the capacity of this arc, but on the existence and values of the travel time on the alternative routes. If there is no such an alternative route, the queue may grow up to infinity and we conclude that a stationary regime for this level of demand flow is impossible. If such an alternative route exists, the equilibrium solution is interpreted as the stationary regime of a dynamic process.

The approach introduced in this paper differs from the traditional practice. To our knowledge, the main problem that attracts the attention of researchers is the description of the congestion induced by commuters during the peak hours. In order to explain this phenomenon for private transportation, one can apply either the Beckmann-type static models (however, see the criticism in Section 3.3), or a dynamic model (such as METROPOLIS, see de Palma and Marchal (2001), DYNASMART, see Mahmassani (1998) or dynaMIT, see Ben-Akiva et al. (1998)). Stable Dynamics proposes a third alternative. Clearly, congestion during the peak hours can hardly be seen as a stationary regime. Therefore, Stable Dynamics can only provide an *aggregate* description of this phenomenon. At the same time, its natural application is the description of the congestion during the *whole day* . From our observations, we know that in large cities, congestion occurs during the whole day and is quite stable. Note that the social effect of congestion during the whole day is much larger than that occurring during the peak hours. For example, traffic count data show that the number of drivers (commuters) who cross the ring roads in large cities during the peak hours is quite small (60 thousands in two directions for Paris Bd. Périphérique, for example). Thus, the social cost of the commuter trips is rather small, and the number of commuter trips tends to decrease from year to year. Contrary, during the day period, there is a lot of commercial traffic with a much higher cost (generalized travel time cost, gasoline, parking cost, wage of the driver, etc.). Therefore, any improvement of transportation networks for daily traffic will immediately result in large social savings. And it seems that the Stable Dynamics approach fits very well such situations.

A major advantage of static over dynamic models is that the former have less data requirements (with the exception of METROPOLIS de Palma and Marchal (2001), which basically uses static data). This means that most dynamic models (micro-simulators, operations planning or planning tools) can hardly be used with only the standard data provided for example by GIS. The collection of simple data (network, number of lanes, jobs and employment) is usually performed in urban areas. This data is exactly the same as those required by Stable Dynamics and as a consequence we believe that such models could play an important role in the near future.

Another advantage of our formulation is that it is based on a few logical rules, which have a simple interpretation. We believe that there is a niche between sophisticated static assignment models, as used by engineers and transportation planners, and over-simplified models as those used by transport economists. Our presentation was mainly motivated by simplicity and consistency. Many extensions could be integrated in the proposed framework. We only presented the main template that will be enriched later on.

To conclude, we want to stress out that the numerical complexity of the models described in this paper is of same level as that one involved in the standard Beckmann models. The latter models can be solved quite efficiently by existing software and we do not expect any numerical difficulties for the new models described in this paper.

Appendix 1: Beckmann model for public transportation

It is clear that the flows of the passengers in the public transportation networks cannot be explained by the models used for private transportation. The main difference is that the travel time in a public network is *constant* and it *does not depend* on the passengers' flows. So, either we assume that the passengers always travel along the shortest path, or we need to find a reason for diversification of the used routes, which is unrelated to travel time. In the sequel, we suggest a criterion for the route choice which takes into account not only the travel time, but also the *convenience* of the trip.

Let us try to formalize the idea of convenience. In order to be more precise, we consider a metropolitan public transport network. The main component of such a network is a *line*, with several stations (or nodes) sequentially connected by directed arcs. For each arc α, we know the travel time $\bar{t}^{(\alpha)} > 0$. We assume that the time interval between the successive trains on this line is constant and equal to $\gamma > 0$ and that each train is composed by the same number of carriages q. Let at some stationary regime we observe a constant flow of passengers $f^{(\alpha)}$ on the arc α. Then the average number of passengers in each carriage is $n_\alpha = \gamma f^{(\alpha)}/q$.

Our main assumption is as follows:

The inconvenience *of travelling by arc α is a* non-decreasing *function of n_α.*

We can find different reasons for such inconvenience. For example, we can assume that inconvenience is equal to zero for $n_\alpha \leq \hat{n}$, where $\hat{n}$ is the number of seats in the carriage. When n_α passes the value $\hat{n}$, we can assume that this characteristics jumps up to a certain level. Then, it increases and goes to infinity as n_α approaches $\bar{n}$, the maximal possible number of passengers in the carriage.

In order to form the cost function of the above arc, we need to express somehow the *inconvenience* in the time units. Then, in view of the above considerations, we can assume that the cost function of the arc α is as follows:

$$c^{(\alpha)}\big(f^{(\alpha)}\big) = \bar{t}^{(\alpha)} + \xi^{(\alpha)}\left(\frac{\gamma}{q} \cdot f^{(\alpha)}\right),$$

where $\xi^{(\alpha)}(\cdot)$ is a non-decreasing function of one variable, $\xi^{(\alpha)}(0) = 0$. Thus, we get an arc performance model, which perfectly fits the Beckmann assumptions.

As soon as we have chosen the functions $\xi^{(\alpha)}(\cdot)$, the remaining part of the model for the metro transportation is quite straightforward. We form the set of origins and the set of destinations and define the O-D matrix. We model all metro lines available in the city and connect some stations by the arcs with fixed travel time (line change). We connect also the nodes of the different lines which share physically the same platform. Finally, we define the access time from the centroids to some stations. In the latter objects we can include also some estimates for the waiting time. As a result, we get a Beckmann model, which can be solved using its dual form (11).

In the dual form of the Beckmann model we work with conjugate functions $\sigma_*^{(\alpha)}(\cdot)$. Let us show by an example how they can be computed from a particular form of the function $\xi^{(\alpha)}(\cdot)$. Let us choose the latter function as follows:

$$\begin{aligned}
&\xi^{(\alpha)}(n) = 0, \quad \text{for } 0 \le n \le \hat{n},\\
&\xi^{(\alpha)}(n) = \frac{a(\bar{n} - \hat{n})}{\bar{n} - n}, \quad \text{for } \hat{n} < n < \bar{n}.
\end{aligned}$$

In this expression a is the inconvenience from absence of free seats, expressed in time units. This inconvenience goes to infinity as n approaches $\bar{n}$, the total capacity of the carriage. Then

$$\sigma^{(\alpha)}(u) = \int_0^u \left[\bar{t}^{(\alpha)} + \xi^{(\alpha)}\left(\frac{\gamma}{q} \cdot \tau\right)\right] d\tau = \bar{t}^{(\alpha)} \cdot u + \frac{aq}{\gamma}(\bar{n} - \hat{n}) \cdot \left(\ln \frac{\bar{n} - \hat{n}}{\bar{n} - \frac{\gamma}{q} u}\right)_+,$$

where $(v)_+ = \max\{0, v\}$. Therefore

$$\begin{aligned}
\sigma_*^{(\alpha)}(\tau) &= \sup_u \left\{\tau u - \sigma^{(\alpha)}(u) : u < \frac{q}{\gamma}\bar{n}\right\}\\
&= \frac{aq}{\gamma}\psi\left(\frac{\tau - \bar{t}^{(\alpha)}}{a}\right), \qquad \operatorname{dom} \sigma_*^{(\alpha)} = [\bar{t}^{(\alpha)}, \infty),
\end{aligned}$$

where

$$\begin{aligned}
&\psi(v) = \hat{n} \cdot v, && \text{for } 0 \le v \le 1,\\
&\psi(v) = \bar{n} \cdot v + (\bar{n} - \hat{n})\left[\ln \frac{1}{v} - 1\right], && \text{for } v \ge 1.
\end{aligned}$$

Note that this function is convex and continuously differentiable.

The above example is only one possibility among many others. Another interesting dependence has the following form:

$$\begin{aligned}
&\xi^{(\alpha)}(n) = 0, && \text{for } 0 \le n \le \hat{n},\\
&\xi^{(\alpha)}(n) = \frac{a(\bar{n} - \hat{n})}{\bar{n} - n} - \frac{a(n - \hat{n})}{\bar{n} - \hat{n}}, && \text{for } \hat{n} < n < \bar{n}.
\end{aligned}$$

In this case, the second expression has zero derivative at $n = \hat{n}$, which looks quite reasonable. Note that the corresponding dual functions for the latter example are also computable in a closed form. In both cases the corresponding equilibrium solution can be easily computed numerically from the dual Beckmann formulation (11). We leave the corresponding modeling details (connections, price of the tickets, etc.) as an exercise for the reader.

Acknowledgments

We would like to thank PREDIT, Ministry of Transportation, France (project QUATUOR) for partial financial support. We also greatly benefited from the stimulating remarks and suggestions of M. Carey, D. Watling and the referees on this paper.

References

Arnott, R., K. Arrow, A. Atkinson, and J. Drèze (eds.) (1994). *Public Economics*, *William Vickrey*. Cambridge, UK: Cambridge University Press.

Anderson, S., A. de Palma, and J.-F. Thisse. (1992). *A Discrete Choice Theory of Product Differentiation*. Cambridge, MA: The MIT Press.

Beckmann, M., B. McGuire, and C. Winsten. (1956). *Studies in the Economics of Transportation*. New Haven, CT: Yale University Press.

Ben-Akiva, M. and S. Lerman. (1985). *Discrete Choice Analysis: Theory and Application to Predict Travel Demand*. Cambridge, MA: The MIT Press.

Ben-Akiva M., M. Bierlaire, H. Koutsopoulos, and R. Mishalani. (1998). DynaMIT: A simulation-based system for traffic prediction and guidance generation. Paper presented at *TRISTAN III*, San Juan, Porto Rico.

Chen, A., D.-H. Lee, and Y. Nie. (2000). "Path and Link Based Traffic Assignment Algorithms: A Comprehensive Computational Study." Paper presented at the *6th ASCE International Conference on Application of Advanced Technologies in Transportation Engineering*, Singapore.

Dafermos, S. (1972). "The Traffic Assignment Problem for Multi-User Transportation Network." *Transportation Science* 6, 73–87.

Dafermos, S. (1980). "Traffic Equilibrium and Variational Inequalities." *Transportation Science* 14, 42–54.

Daganzo, C. and Y. Sheffi. (1977). "On Stochastic Models of Traffic." *Transportation Science* 11, 253–274.

de Palma, A. and F. Marchal. (2001). "Dynamic Traffic Analysis with Static Data: Some Guidelines with an Application to Paris." Paper presented at the *80th Annual Meeting* TRB Meeting of the *Transportation Record Board*, Washington, DC.

Der-Hong, Lee and Yu N. (2001). "Accelerating Strategies and Computational Studies of the Frank-Wolfe Algorithm for the Traffic Assignment Problem." Paper presented at the *80th Annual Meeting TRB Meeting of the Transportation Record Board*, Washington, DC.

Florian, M. (1977). "A Traffic Equilibrium Model of Travel by Car and Pubic Transit Modes." *Transportation Science* 11, 166–179.

Frank, M. and P. Wolfe. (1956). "An Algorithm for Quadratic Programming." *Naval Research Logistics.* Quarterly 3, 95–110.

Knuth, D.E. (1979). *The Art of Computer Programming*. Addison-Wesley.

Larson, T. and M. Patrickson. (1997). "Side Constrained Traffic Equilibrium Models—Analysis, Computation and Applications." Department of Mathematics, Linkoping Institute of Technology, Sweden.

LeBlanc, L., R. Helgason, and D. Boyce. (1985). "Improved Efficiency of the Frank-Wolfe Algorithm for Convex Network Problems." *Transportation Science* 19, 445–462.

Lévy-Lambert, H. (1968). "Tarification Des Services à Qualité Variable: Application Aux Péages De Circulation," *Econometrica* 36, 564–575.

Mahmassani, H. (1998). "Development of DYNASMART-X Software for Real-Time Dynamic Traffic Assignment," Technical Report ST067-85-TASK E, Center for Transportation, The University of Texas at Austin.

Nagurney, A. (1993a). *Network Economics: A Variational Inequality Approach*. Dordrecht: Kluwer Academic Publishers.

Nagurney, A. (1993b). *Network Economics: A Variational Inequality Approach*. Norwell, MA: Kluwer Academic Publisher.

Nemirovsky, A. and D.Yudin. (1983). *Informational Complexity and Efficient Methods for Solution of Convex Extremal Problems*. New York: Wiley.

Nesterov, Yu. (2000). "Stable Traffic Equilibria: Properties and Applications." *Optimization and Engineering* 3, 29–50.

Nesterov, Yu. and A. de Palma. (2000). "Stable Dynamics in Transportation Systems." CORE DP 2000/27, Belgium: Catholic University of Louvain.

Patrickson, M. (1994). *The Traffic Assignment Problem: Models and Methods*. Utrechet.

Sheffi, Y. (1985). *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Englewood Cliffs, NJ: Prentice-Hall.

Smith, M.J. (1979). "Existence, Uniqueness and Stability of Traffic Equilibria." *Transportation Research* B, 295–304.

Steinberg, R. and R. Stone. (1988). "The Prevalence of Paradoxes in Transportation Equilibrium Problems." *Transportation Science* 22, 231–241.

Wardrop, J. (1952). "Some Theoretical Aspects of Road Traffic Research." *Proceedings of the Institute of Civil Engineers* Part II, 1, 325–378.

Yang, H and W. Lam. (1996). "Optimal Roads Tolls Under Conditions of Queuing and Congestion." *Transportation Research* 30A, 319–332.