THE TOPOLOGY OF SUBSETS OF $\mathbb{R}^n$

The basic material of this lecture should be familiar to you from Advanced Calculus courses, but we shall revise it in detail to ensure that you are comfortable with its main notions (the notions of open set and continuous map) and know how to work with them.

1.1. Continuous maps

"Topology is the mathematics of continuity"

Let $\mathbb{R}$ be the set of real numbers. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *continuous at the point* $x_0 \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality

$$|f(x_0) - f(x)| < \varepsilon$$

holds for all $x \in \mathbb{R}$ whenever $|x_0 - x| < \delta$. The function f is called *continuous* if it is continuous at all points $x \in \mathbb{R}$.

This is basic one-variable calculus.

Let $\mathbb{R}^n$ be n-dimensional space. By $O_r(p)$ denote the *open ball* of radius $r > 0$ and center $p \in \mathbb{R}^n$, i.e., the set

$$O_r(p) := \{q \in \mathbb{R}^n \mid d(p,q) < r\},$$

where d is the distance in $\mathbb{R}^n$. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *continuous at the point* $p_0 \in \mathbb{R}^n$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(p) \in O_\varepsilon(\ f(p_0))$ for all $p \in O_\delta(p_0)$. The function f is called *continuous* if it is continuous at all points $p \in \mathbb{R}^n$.

This is (more advanced) calculus in several variables.

A set $G \subset \mathbb{R}^n$ is called *open in* $\mathbb{R}^n$ if for any point $g \in G$ there exists a $\delta > 0$ such that $O_\delta(g) \subset G$. Let $X \subset \mathbb{R}^n$. A subset $U \subset X$ is called *open in* X if for any point $u \in U$ there exists a $\delta > 0$ such that $O_\delta(u) \cap X \subset U$. An equivalent property: $U = V \cap X$, where V is an open set in $\mathbb{R}^n$. Clearly, any union of open sets is open and any finite intersection of open sets is open. Let X and Y be subsets of $\mathbb{R}^n$. A map $f\colon X \to Y$ is called *continuous* if the preimage of any open set is an open set, i.e.,

$$V \text{ is open in } Y \implies f^{-1}(V) \text{ is open in } X.$$

This is basic topology.

Let us compare the three definitions of continuity. Clearly, the topological definition is not only the shortest, but is conceptually the simplest. Also, the topological definition yields the simplest proofs. Here is an example.

Theorem 1.1. *The composition of continuous maps is a continuous map. In more detail, if X, Y, Z are subsets of $\mathbb{R}^n$, $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then their composition, i.e., the map $h = g \circ f\colon X \to Z$ given by $h(x) := g(\,f(x))$, is continuous.*

Proof. Let $W \subset Z$ be open. Then the set $V := f^{-1}(W) \subset Y$ is open (because f is continuous). Therefore, the set $U := g^{-1}(V) \subset X$ is open (because g is continuous). But $U = h^{-1}(W)$. □

Compare this proof with the proof of the corresponding theorem in basic calculus. This proof is much simpler.

The notion of open set, used to define continuity, is fundamental in topology. Other basic notions (neighborhood, closed set, closure, interior, boundary, compactness, path connectedness, etc.) are defined by using open sets.

1.2. Closure, boundary, interior

By a *neighborhood* of a point $x \in X \subset \mathbb{R}^n$ we mean any open set (in X) that contains x.

Let $A \subset X$; an *interior point* of A is a point $x \in A$ which has a neighborhood U in X contained in A. The set of all interior points of A is called the *interior* of A in X and is denoted by $\mathrm{Int}(A)$. An *isolated point* of A in X is a point $a \in A$ which has a neighborhood U in X such that $U \cap A = a$.

A *boundary point* of A in X is a point $x \in X$ such that any neighborhood $U \ni x$ in X contains points of A and points not in A, i.e., $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$; the boundary of A is denoted by $\mathrm{Bd}(A)$ or ∂A. The union of A and all the boundary points of A is called the *closure* of A in X and is denoted by $\mathrm{Clos}(A, X)$ (or $\mathrm{Clos}(A)$, or $\overline{A}$, if X is clear from the context).

Theorem 1.2. Let $A \subset \mathbb{R}^n$.

(a) *A is closed if and only if it contains all of its boundary points.*

(b) *The interior of A is the largest (by inclusion) open set contained in A.*

(c) *The closure of A is the smallest (by inclusion) closed set containing A.*

(d) *The boundary of a set A is the difference between the closure of A and the interior of A: $\mathrm{Bd}(A) = \mathrm{Clos}(A) - \mathrm{Int}(A)$.*

The proofs is follow directly from the definitions, and you should remember them from the Calculus course. You be able to write them up without much trouble in the exercise class.

1.3. Topological equivalence

"A topologist is person who can't tell the difference between a coffee cup and a doughnut."

The goal of this subsection is to teach you to visualize objects (geometric figures) the way topologists see them, i.e., by regarding figures as equivalent if they can be bijectively deformed into each other. This is something you have not been taught to do in calculus courses, and it may take you some time before you will become able to do it.

Let X and Y be "geometric figures," i.e., arbitrary subsets of $\mathbb{R}^n$. Then X and Y are called *topologically equivalent* or *homeomorphic* if there exists a *homeomorphism* of X onto Y, i.e., a continuous bijective map $h\colon X \to Y$ such that the inverse map h^{-1} is continuous.

For the topologist, homeomorphic figures are the same figure: a circle is the same as (the boundary of) a square, or a triangle, or a hexagon, or an ellipse; an arc of a circle is the same as a closed interval, a 2-dimensional disk is the same as one of its segments or as a triangle together with its inner points; the boundary of a cube is the same as a sphere, or as (the boundary of) a cylinder, or (the boundary of) a tetrahedron.

If a property does not change under any homeomorphism, then this property is called *topological*. Examples of topological properties are compactness and path connectedness (they will be defined later in this lecture). Examples of properties that are *not* topological are length, area, volume, and boundedness. The fact that boundedness is not a topological property may seem rather surprising; as an illustration, we shall prove that

the open interval $(0,1)$ *is homeomorphic to the real line* $\mathbb{R}$ (!)

This is proved by constructing an explicit homeomorphism $h : (0,1) \to \mathbb{R}$ as the composition of the two homeomorphisms p and s shown in Figure 1.1.

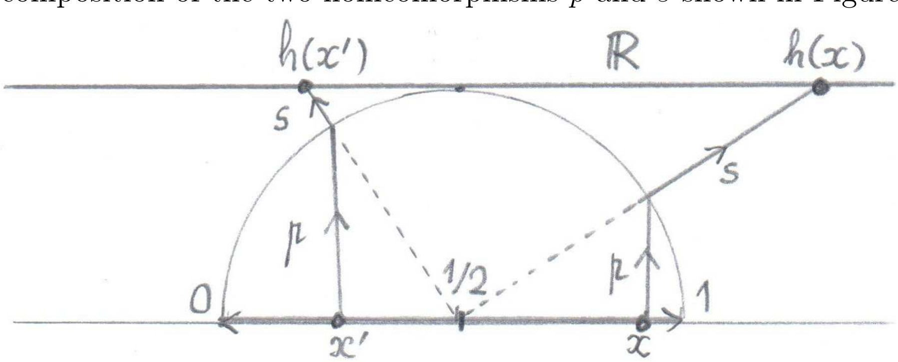

FIGURE 1.1. The homeomorphism $h : (0,1) \to \mathbb{R}$

For another illustration, look at Figure 1.2; you should intuitively feel that the torus is not homeomorphic to the sphere (although we are at present unable to prove this!). However, the ordinary torus *is* homeomorphic to the knotted torus in the figure, although they look "topologically very different"; they provide examples of figures that are homeomorphic, but are embedded in $\mathbb{R}^3$ in different ways. We shall come back to this distinction later in the course, in particular in the lecture on knot theory.

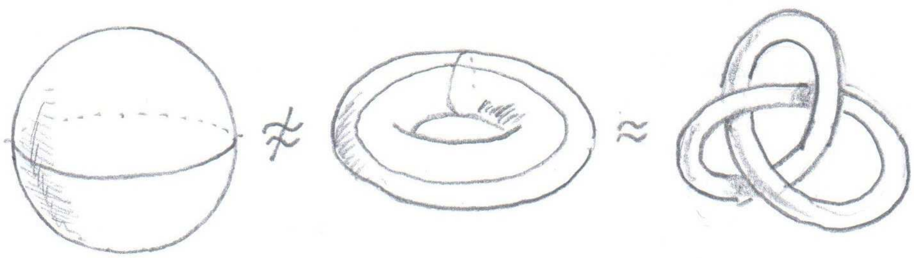

FIGURE 1.2. The sphere and two tori

We conclude this lecture by studying two basic topological properties of geometric figures that will be constantly used in this course.

1.4. Path connectedness

A set $X \subset \mathbb{R}^n$ is called *path connected* if any two points of X can be joined by a path, i.e., if for any $x, y \in X$ there exists a continuous map $\varphi\colon [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

Theorem 1.3. *The continuous image of a path connected set is path connected. In more detail, if the map $f\colon X \to Y$ is continuous and X is path connected, then $f(X)$ is path connected.*

Proof. Let $y_1,\ y_2 \in f(X)$. Let $X_1 := f^{-1}(y_1)$ and $X_2 := f^{-1}(y_2)$. Let x_1 and x_2 be arbitrary points of X_1 and X_2, respectively. Then there exists a continuous map $\varphi\colon [0,1] \to X$ such that $\varphi(0) = x_1$ and $\varphi(1) = x_2$ (because X is path connected). Let $\psi\colon [0,1] \to f(X)$ be defined by $\psi := f \circ \varphi$. Then ψ is continuous (by Theorem 1.1), $\psi(0) = y_1$ and $\psi(1) = y_2$. □

Thus we have shown that path connectedness is a topological property.

1.5. Compactness

A family $\{U_\alpha\}$ of open sets in $X \subset \mathbb{R}^n$ is called an *open cover* of X if this family covers X, i.e., if $\cup_\alpha U_\alpha \supset X$. A *subcover* of $\{U_\alpha\}$ is a subfamily $\{U_{\alpha_\beta}\}$ such that $\cup_\beta U_{\alpha_\beta} \supset X$, i.e., the subfamily also covers X. The set X is called *compact* if every open cover of X contains a finite subcover.

Note the importance of the word "every" in the last definition: a set in noncompact if *at least one* of its open covers contains no finite subcover of X. As an illustration, let us show that

the open interval $(0,1)$ *is not compact.*

Indeed, this follows from the fact that any finite subfamily of the cover $\{U_1, U_2, \dots\}$ shown in Figure 1.3 obviously does not cover $(0,1)$.

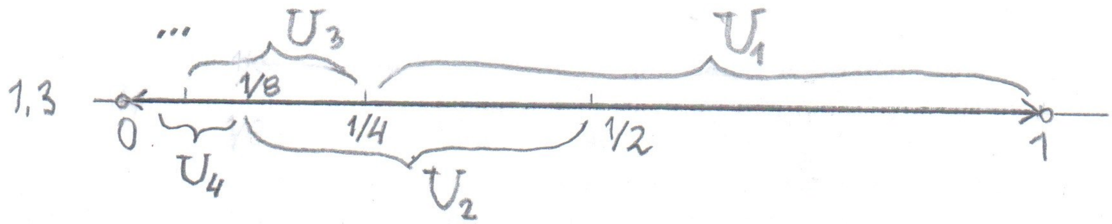

FIGURE 1.3. The open interval $(0,1)$ is not compact

Theorem 1.4. *The continuous image of a compact set is compact, i.e., if a map* $f\colon X \to Y$ *is continuous and* $X \subset \mathbb{R}^n$ *is compact, then* $f(X)$ *is compact.*

Proof. Let $\{V_\alpha$ be an open covering of $f(X)$. Then each $U_\alpha ;= f^{-1}(V_\alpha)$ is open in X (by the definition of continuity) and so $\{U_\alpha\}$ is an open covering of X. But X is compact, hence $\{U_\alpha\}$ has a finite subcovering, say $\{U_{\alpha_1}, \dots, U_{\alpha_N}\}$. Then $\{f(U_{\alpha_1}), \dots, f(U_{\alpha_N})\}$ is obviously a finite subcover of $\{V_\alpha\}$. □

Thus we have shown that compactness is a topological property.

Fact 1.5. *A set* $X \subset \mathbb{R}^n$ *is compact if and only if* X *is closed and bounded.*

We do not give the proof of this fact because it not really topological: the word "bounded" makes no sense to a topologist; the proof is usually given in calculus courses.

1.6. Exercises

1.1. Using the ε–δ definition of continuity, give a detailed proof of the fact that the composition of two continuous functions is continuous.

1.2. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$. Suppose the functions $f_{1,x_0}(y) := F(x_0, y)$ and $f_{2,y_0}(x) := F(x, y_0)$ are continues for any x_0, $y_0 \in \mathbb{R}$. Is it true that $F(x, y)$ is continuous?

1.3. Prove the four assertions (a)-(d) of Theorem 1.1.

1.4. The towns A and B are connected by two roads. Two travellers can walk along these roads from A to B so that the distance between them at any moment is less than or equal to 1 km. Can one traveller walk from A to B and the other from B to A (using these roads) so that the distance between them at any moment is greater than 1 km?

1.5. Suppose $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The *distance* from the point x to the subset A is equal to $d(x, A) = \inf\{\|x - a\| : a \in A\}$.

(i) Prove that the function $f(x) = d(x, A)$ is continuous for any $A \subset \mathbb{R}^n$.

(ii) Prove that if the set A is closed, then the function $f(x) = d(x, A)$ is positive for any $x \notin A$.

1.6. Let X be the subset of $\mathbb{R}^2$ given by the equation $xy = 0$ (X is the union of two lines). Give some examples of neighborhoods: (a) of the point $(0, 0)$; (b) of the point $(0, 1)$.

1.7. Describe the set of points x in $\mathbb{R}^2$ such that $d(x, A) = 1$; 2; 3, where the set A is given by the formula:

(a) $x^2 + y^2 = 0$; (b) $x^2 + y^2 = 2$;
(c)* $x^2 + 2y^2 = 2$; (d) the square of area two.

1.8. Let A and B be two subsets of the set X that was defined in Exercise 1.7. Suppose that A and B are homeomorphic and A is open in X. Is it true that B is also open in X?

1.9. Construct a homeomorphism between the boundary of the cube $\mathbb{I}^3$ and the sphere $\mathbb{S}^2$.

1.10. Construct a homeomorphism between the plane $\mathbb{R}^2$ and the open disk $\mathbb{B}^2 := \{v \in \mathbb{R}^2 : |v| < 1\}$.

1.11. Construct a homeomorphism between the plane $\mathbb{R}^2$ and the sphere $\mathbb{S}^2$ with one point removed.