Lecture 11

COVERING SPACES

A covering space (or covering) is not a space, but a mapping of spaces (usually manifolds) which, locally, is a homeomorphism, but globally may be quite complicated. The simplest nontrivial example is the exponential map $\mathbb{R} \to \mathbb{S}^1$ discussed in Lecture 6.

11.1. Definition and examples

In this lecture, we will consider only path connected spaces with basepoint and only basepoint-preserving maps. Suppose E, B are path connected topological spaces $p\colon E \to B$ is a continuous map such that $p^{-1}(y)$ is a discrete subspace, the cardinality of the set $p^{-1}(y) := D$ is independent of $y \in B$ and every $x \in p^{-1}(y)$ has a neighborhood on which p is a homeomorphism to a neighborhood of $y \in B$, then the quadruple (p, T, B, D) is called a *covering* with *covering projection* p, *total space* E, *base* B, and *fiber* $D = p^{-1}(y)$.

If $n = |D|$ is finite, then (p, E, B, D) is said to be an *n-fold covering*. If D is countably infinite (счетное in Russian), we say that $p : E \to B$ is a *countable* covering.

Examples 11.1: (i) the map $w_3 : \mathbb{S}^1 \to \mathbb{S}^1$, given by $e^{i\varphi} \mapsto e^{i3\varphi}$ is a 3-fold covering of the circle by the circle;

(ii) the exponential map exp:$\mathbb{R} \to \mathbb{S}^1$ is a countable covering of the circle by the real line;

(iii) the map $u : \mathbb{R}^2 \to \mathbb{T}^2$, $(x, y) \mapsto (2\pi\{x\}, 2\pi\{y\})$, where $\{\cdot\}$ denotes the fractional part of a real number, is a countable covering of the torus by the plane;

(iv) the map $\tau : \mathbb{S}^2 \to \mathbb{R}P^2$ obtained by identifying antipodal points of the sphere is a 2-fold covering of the projective plane.

Like any other important class of mathematical objects, covering spaces form a category. In this category, a morphism between two covering spaces $p_i : E_i \to B_i$, $i = 1, 2$, are pairs of (continuous, basepoint-preserving) maps $\phi : B_1 \to B_2$ and $\Phi : E_1 \to E_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{\phi} & B_2 \end{array}$$

Compositions of morphisms and identical morphisms are defined in the natural way. Then, obviously, an isomorphism of covering spaces is a morphism for which Φ and ϕ are homeomorphisms. Isomorphic covering spaces are considered identical.

If E is simply connected, then the covering $p : E \to B$ is called *universal*.

If $f\colon X \to B$ is continuous and $\widetilde{f}\colon X \to E$ satisfies $f = p \circ F$, then $\widetilde{f}$ is said to be a *lift* of f. If $f\colon B \to B$ is continuous and $\widetilde{f}\colon E \to E$ is continuous and satisfies $f \circ p = pi \circ \widetilde{f}$, then $\widetilde{f}$ is said to be a *lift* of f as well. The figure below shows the lift of a closed curve.

A homeomorphism of the total space of a covering E of E is called a *deck transformation* ("монодромия" in Russian), if it is a lift of the identity on B.

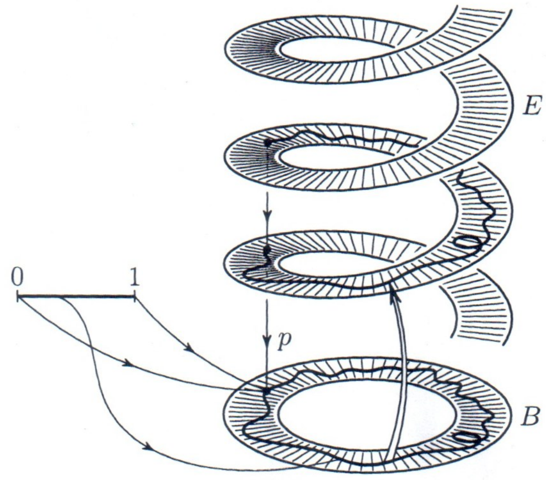

FIGURE 10.2. Lift of a closed curve

11.2. Path lifting and covering homotopy

In this section, we prove two important technical assertions which allow, given a covering space $p : E \to B$, to lift "upstairs" (i.e., to E) continuous processes taking place "downstairs" (i.e., in B). The underlying idea has already been exploited when we defined the degree of circle maps by using the exponential map (see Lecture??), and we will now be generalizing the setting of the exponential map to arbitrary covering spaces.

Lemma 11.1. [Path lifting lemma] *Any path in the base B of a covering space $p : E \to B$ can be lifted to the total space of the covering, and the lift is unique if its initial point in the covering is specified. More precisely, if $p : E \to B$ is a covering space, $\alpha : [0,1] \to B$ is any path, and $x_0 \in p^{-1}(\alpha(0))$, then there exists a unique map $\widetilde{\alpha} : [0,1] \to X$ such that $p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0) = x_0$.*

Proof. By the definition of covering space, for each point $b \in \alpha([0,1]))$ there is a neighborhood U_b whose inverse image under p falls apart into disjoint neighborhoods each of which is projected homeomorphically by p onto U_b. The set of all such U_b covers $\alpha([0,1])$ and, since $\alpha([0,1])$ is compact, it possesses a finite subcover that we denote by $U_0, U_1, \dots U_k$.

Without loss of generality, we assume that U_0 contains $b_0 := \alpha(0)$ and denote by $\widetilde{U}_0$ the component of $p^{-1}(U_0)$ that contains the point x_0. Then we can lift a part of the path α contained in U_0 to $\widetilde{U}_0$ (uniquely!) by means of the inverse to the homeomorphism between $\widetilde{U}_0$ and U_0.

Now, again without loss of generality, we assume that U_1 intersects U_0 and contains points of $\alpha[0,1]$ not lying in U_0. Let $b_1 \in \alpha([0,1])$ be a point contained both in U_0 and U_1 and denote by $\widetilde{b}_1$ the image of b_1 under $p^{-1}\big|_{U_0}$. Let $\widetilde{U}_1$ be the component of the inverse image of U_1 containing $\widetilde{b}_1$. We now extend the lift of our path to its part contained in U_1 by using the inverse of the homeomorphism between $\widetilde{U}_1$ and U_1. Note that the lift obtained is the only possible one. Our construction in the case when the path is closed (i.t., is a loop) is shown in Figure 11.1.

Continuing in this way, after a finite number of steps we will have lifted the entire path $\alpha([0,1])$ to X, and the lift obtained will be the only one obeying the conditions of the lemma. □

Remark 11.1. Note that the lift of a closed path is *not* necessarily a closed path, as we have already seen in our discussion of the degree of circle maps.

Note also that if all paths (i.e., maps of $A = [0,1]$) can be lifted, it is not true that all maps of *any* space A can be lifted (see Exercise ??).

Now we generalize the path lifting lemma to homotopies, having in mind that a path is actually a homotopy, namely a homotopy of the one-point space. This trivial observation is not only the starting point of the formulation of the covering homotopy theorem, but also the key argument in its proof.

Theorem 11.2. [Covering homotopy theorem] *Any homotopy in the base of a covering space can be lifted to the covering, and the homotopy is unique if its initial map in the covering is specified as a lift of the initial map of the given homotopy. More precisely, if $p : E \to B$ is a covering, $F : A \times [0,1] \to B$ is any homotopy whose initial map $f_0(\cdot) := F(\cdot, 0)$ possesses a lift $\widetilde{f}_0$, then there exists a unique homotopy $\widetilde{F} : A \times [0,1] \to X$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F}(\cdot, 0) = \widetilde{f}_0(\cdot)$.*

Proof. The theorem will be proved by reducing the theorem to the path lifting lemma from the previous subsection.Fix some point $\alpha \in A$. Define $\alpha_a(t); = F(a,t)$ and denote by x_a the point $\widetilde{f}_0(a)$. Then α_a is a path, and by the path lifting lemma,there exists a unique lift $\widetilde{\alpha}_a$ of this path such that $\widetilde{\alpha}(0) = x_a$. Now consider the homotopy defined by

$$\widetilde{F}(a,t) := \widetilde{\alpha}_a(t), \quad \text{for all} \quad a \in A, \quad t \in [0,1].$$

Then, we claim that $\widetilde{F}$ satisfies all the conditions of the theorem, i.e., $\widetilde{F}$ is continuous and unique. We leave this verification to the reader. □

Remark 11.2. The covering homotopy theorem is not true if $E \to B$ is an arbitrary surjection (and not a covering space). For a counterexample, see Exercise 7.

11.3. Role of the fundamental group

The projection p of a covering space $p : E \to B$ induces a homomorphism $p_\# : \pi_1(E) \to \pi_1(B)$. We will see that when the spaces E and B are "locally nice", the homomorphism $p_\#$ entirely determines (up to isomorphism) the covering space p over a given B. (What we mean by "locally nice" will be explained below.)

More precisely, in this section we will show that, provided that the "local nicety" condition holds, $p_\#$ is a monomorphism and that, given a subgroup G of $\pi_1(B)$, we can effectively construct a unique space E and a unique (up to isomorphism) covering map

$p : E \to B$ for which G is the image of $\pi_1(E)$ under $p_\#$. Moreover, we will prove that there is a bijection between conjugacy classes of subgroups of $\pi_1(B)$ and isomorphism classes of coverings, thus achieving the classification of all coverings over a given base B in terms of $\pi_1(B)$.

Theorem 11.2 *The homomorphism $p_\# : \pi_1(E) \to \pi_1(B)$ induced by any covering space $p : E \to B$ is a monomorphism.*

Proof. The theorem is an immediate consequence of the homotopy lifting property proved in the previous section. Indeed, it suffices to prove that a nonzero element $[\alpha]$ of $\pi_1(E)$ cannot be taken to zero by $p_\#$. Assume that $p_\#([\alpha]) = 0$. This means that the loop $p \circ \alpha$, where $\alpha \in [\alpha]$, is homotopic to a point in B. By the homotopy lifting theorem, we can lift this homotopy to E, which means that $[\alpha] = 0$. □

Now we describe the main construction of this lecture: given a space and a subgroup of its fundamental group, we will construct the corresponding covering. This construction works provided the space considered is "locally nice" in a sense that will be specified below.

Theorem 11.3. *For any "locally nice" space B and any subgroup $G \subset \pi_1(B, b_0)$, there exists a unique covering space $p : X \to B$ such that $p_\#(X) = G$.*

Proof. The theorem is proved by means of another trick. Let us consider the set $P(B, b_0)$ of all paths in B issuing from b_0. Two paths $\alpha_i : [0,1] \to B$, $i = 1, 2$ will be identified (notation $\alpha_1 \sim \alpha_2$) if they have a common endpoint and the loop λ given by

$$\lambda(t) = \begin{cases} \alpha_1(2t) & \text{if } \ 0 \le t \le 1/2, \\ \alpha_2(2 - 2t) & \text{if } \ 1/2 \le t \le 1. \end{cases}$$

determines an element of $\pi_1(B)$ that belongs to G. (The loop λ can be described as first going along α_1 (at double speed) and then along α_2 from its endpoint back to b_0, also at double speed.)

Denote by $X := P(B, b_0)/_\sim$ the quotient space of $P(B, b_0)$ by the equivalence relation just defined. Endow X with the "natural" topology (the formal definition is given below) and define the map $p : X \to B$ by stipulating that it takes each equivalence class of paths in $P(B, b_0)$ to the endpoint of one of them (there is no ambiguity in this definition, because equivalent paths have the same endpoint).

Then $p : X \to B$ is the required covering space. It remains to:
(o) define the topology on X; (i) prove that p is continuous; (ii) prove that p is a local homeomorphism; (iii) prove that $p_\#(\pi_1(X))$ coincides with G; (iv) prove that p is unique. We will do this after defining what we mean by "locally nice".

Remark 11.4. To understand the main idea of the construction described above, the reader should try applying in the case $G = 0$ (construction of the universal cover).

Remark 11.5. The above construction is not effective at all, and cannot be used to describe the covering space obtained. However, in reasonably simple cases it is easy to guess what the space X is from the fact that the fundamental group of X is G and p is a local homeomorphism.

A topological space X is called *locally path connected* if for any point $x \in X$ and any neighborhood U of x there exists a smaller neighborhood $V \subset U$ of x which is path connected. A topological space X is called *locally simply connected* if for any point $x \in X$ and any neighborhood U of x there exists a smaller neighborhood $V \subset U$ of x which is simply connected.

Examples 11.2. (a) Let $X \subset \mathbb{R}^2$ be the union of the segments

$$\{(x,y) \,|\, y = 1/2^n,\ 0 \le x \le 1\} \quad n = 0,1,2,3,\ldots$$

and the two unit segments $[0,1]$ of the x-axis and y-axis (see Figure 11.2(a)). Then X is path connected but not locally path connected (at all points of the interval $(0,1]$ of the x-axis).

(b) Let $X \subset \mathbb{R}^2$ be the union of the circles

$$\{(x,y) \,|\, x^2 + (y-1/n)^2 = 1/n^2\} \quad n = 1,2,3,\ldots;$$

the circles are all tangent to the x-axis and to each other at the point $(0,0)$ (see Figure 11.2 (b)). Then X is path connected but not locally simply connected (at the point$(0,0)$).

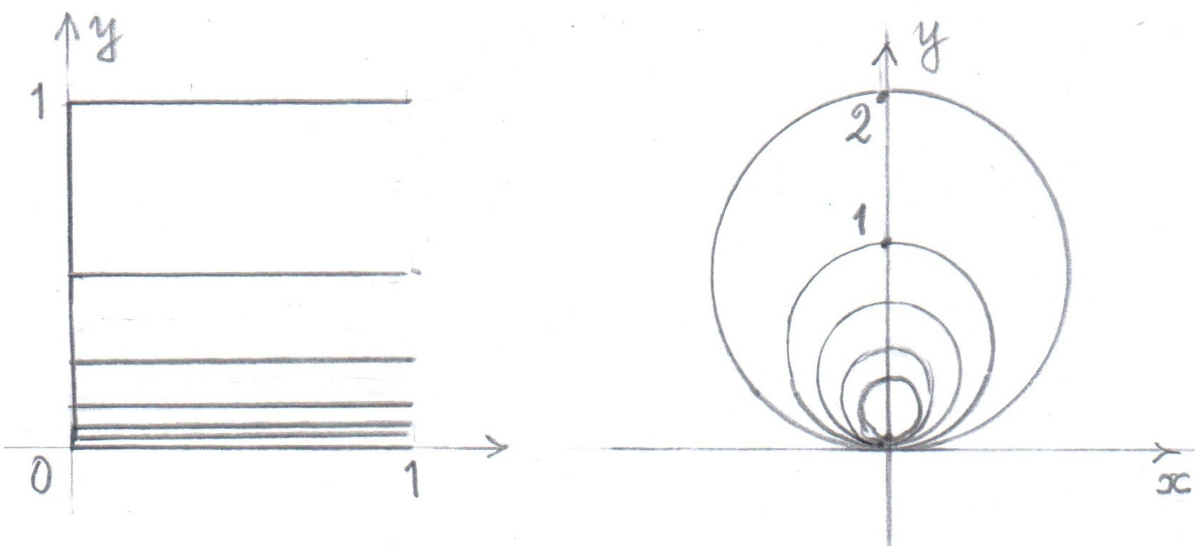

FIGURE 11.2. Not locally connected and not locally simply connected spaces

We will now conclude the proof of Theorem 11.3, assuming that B is locally pathconnected and locally simply connected.

(o) *Definition of the topology in* $X = P(B,b_0)/_\sim$. In order to define the topology, we will specify a base of open sets of rather special form, which will be very convenient for our further considerations. Let U be an open set in B and $x \in X$ be a point such that $p(x) \in U$. Let α be one of the paths in x with initial point x_0 and endpoint x_1. Denote by (U,x) the set of equivalence classes (with respect to $\sim$) of extensions of the path α whose segments beyond x_1 lie entirely inside U. Clearly, (U,x) does not depend on the choice of $\alpha \in x$.

We claim that (U,x) actually does not depend on the choice of the point x in the following sense: if $x_2 \in (U,x_1)$, then $(U,x_1) = (U,x_2)$. To prove this, consider the points $b_1 := p(x_1)$ and $b_2 := p(x_2)$. Join the points b_1 and (b_2) by a path (denoted β) contained in U.

Let $\alpha\alpha_1$ denote an extension of α, with the added path segment α_1 contained in U. Now consider the path $\alpha\beta\beta^{-1}\alpha_1$, which is obviously homotopic to $\alpha\alpha_1$. On the other hand, it may be regarded as the extension (beyond x_2) of the path $\alpha\beta$ by the path $\beta^{-1}\alpha_{-1}$. Therefore, the assignment $\alpha\alpha_1 \mapsto \alpha\beta\beta^{-1}\alpha_1$ determines a bijection between (U,x_1) and (U,x_2), which proves our claim.

Now we can define the topology in X by taking for a base of the topology the family of all sets of the form (U,x). To prove that this defines a topology, we must check that that a nonempty intersection of two elements of the base contains an element of the base. Let the point x belong to the intersection of the sets (U_1,x_1) and (U_2,x_2). Denote $V := U_1 \cap U_2$ and consider the set (V,x); this set is contained in the intersection of the sets (U_1,x_1) and (U_2,x_2) (in fact, coincides with it) and contains x, so that $\{(U,x)\}$ is indeed a base of a topology on X.

(i) *The map p is continuous.* Take $x \in X$. Let U be any path connected and simply connected neighborhood of $p(x)$ (it exists by the condition imposed on B). The inverse image of U under p is consists of basis open sets of the topology of X (see item (o)) and is therefore open, which establishes the continuity at an (arbitrary) point $x \in X$.

(ii) *The map p is a local homeomorphism.* Take any point $x \in X$ and denote by $p|_U : (U,x) \to U$ the restriction of p to any basis neighborhood (U,x) of x, so that U will be an open path connected and simply connected set in B. The path connectedness of U implies the surjectivity of $p|_U$ and its simple connectedness, the injectivity of $p|_U$.

(iii) *The subgroup $p_\#(\pi_1(X))$ coincides with G.* Let α be a loop in B with basepoint b_0 and $\widetilde{\alpha}$ be the lift of α initiating at x_0 ($\widetilde{\alpha}$ is not necessarily a closed path). The subgroup $p_\#(\pi_1)(X)$ consists of homotopy classes of the loops α whose lifts $\widetilde{\alpha}$ are closed paths. By construction, the path $\widetilde{\alpha}$ is closed iff the equivalence class of the loop α corresponds to the point x_0, i.e., if the homotopy class of α is an element of G.

(iv) *The map p is unique.* We omit the proof of this fact here.

11.4. Regular coverings.

A covering $p : E \to B$ is called *regular* if the subgroup $p_\#(\pi_1(E)) \subset \pi_1(B)$ is normal.

Theorem 11.4. *If $p : E \to B$ is any regular covering, then the quotient group $\pi(B)/p_\#(\pi_1(E))$ is isomorphic to the group of deck transformations of the fiber $D = p^{-1}(b_0)$ (here b_0 is the basepoint of B).*

Proof. There is a natural bijection between the right cosets (правые смежные классы in Russian) of the subgroup $p_\#(\pi_1(E)) \subset \pi_1(B)$ and D, but since this subgroup is normal, these cosets forms a group that "shuffles" the points of D, so that the quotient group $\pi(B)/p_\#(\pi_1(E))$ is the group of deck transformations of D.

11.5. Exercises

11.1. Suppose that one surface is covered by another surface. What is the relation between their Euler characteristics, if the covering is n-fold?

11.2. Prove that the sphere with g_1 handles can be covered by the sphere with g_2 handles (g_1, $g_2 \geq 2$) iff $g_1 - 1$ is a divisor of $g_2 - 1$.

11.3. Construct a nonregular covering of the wedge product of two circles.

11.4. Construct two regular coverings of the wedge product of two circles that are not homotopy equivalent to each other.

11.5. Prove that for any $n \geq 2$ the wedge product of two circles can be covered by the wedge product of n circles.

11.6. Prove that if the base surface of a covering $p\colon N^2 \to M^2$ is orientable, then so is the covering surface N^2.

11.7. Let X be the union of the lateral surface of the cone and the half-line issuing from its vertex v, and let $p : X \to B$ be the natural projection of X on the line $B = \mathbb{R}$. Show that $p : X \to B$ does not possess the covering homotopy property.

11.8. Let the covering surface N^2 of a covering $p\colon N^2 \to M^2$ is orientable. Is it true that the base surface M^2 is orientable?

11.9. Can $\mathbb{R}P^2$ cover the sphere?

11.10. Can the torus T^2 cover T^2 by a 3-fold covering?

11.11. Can $\mathbb{R}P^2$ be covered by the plane?

11.12. Construct the universal covering of the Möbius band.

11.13. Construct the universal covering of the torus $\mathbb{T}^3$.

11.14. Construct the universal covering of the wedge product of two circles.

11.15. Construct the universal covering of the wedge product $\mathbb{S}^1 \vee \mathbb{S}^2$.

11.16. Construct the universal covering of the sphere with $g \geq 2$ handles.

11.17. Suppose some connected graph G has e edges and v vertices. Find the fundamental group of the graph G.

11.18. Prove that any subgroup of a free group is a free group.

11.19. Prove that the free group of rank 2 contains as a subgroup the free group of rank n for all n (including $n = \infty$).

11.20. Give an example of a covering space $p : E \to B$, of a space A, and a map $f : A \to B$ that cannot be lifted to E.

11.21. Prove that the universal cover $\omega : U \to B$ of any (pathconnected) space B is the cover of any other covering of B, i.e., for any covering space $p : E \to B$, there exists a covering space $q : U \to E$.