Lecture 11

Knots and Links

Knot theory, which studies knots, links (зацепления in Russian), and their invariants, has a long history that begins at the end of the 18th century (Vandermonde), with significant contributions by Gauss, Poincaré, Reidemeister, Alexander, Conway, Fox, flourished at the end of the 20th century (four Fields medalists – Jones, Witten, Drinfeld, Kontsevich – and Kauffman, Reshetikhin, Turaev, Viro, Vassiliev, Khovanov) and is still going strong today.

12.1. Main definitions

A *knot* is a closed non-self-intersecting broken line in $\mathbb{R}^3$, a *link* is a set of nonintersecting and non-self-intersecting broken lines in $\mathbb{R}^3$. Two knots or links are called *isotopic* if there exists a finite sequence of Δ-moves (see Figure 12.1) transforming one into the other.

Note that in the definition of Δ-moves the knot (link) does not intersect the triangle ABC defining the move except along its sides, as shown in the figure. In the definition of a Δ-move, we include the case when the triangle ABC is degenerate; in that case the move reduces to adding or removing a vertex.

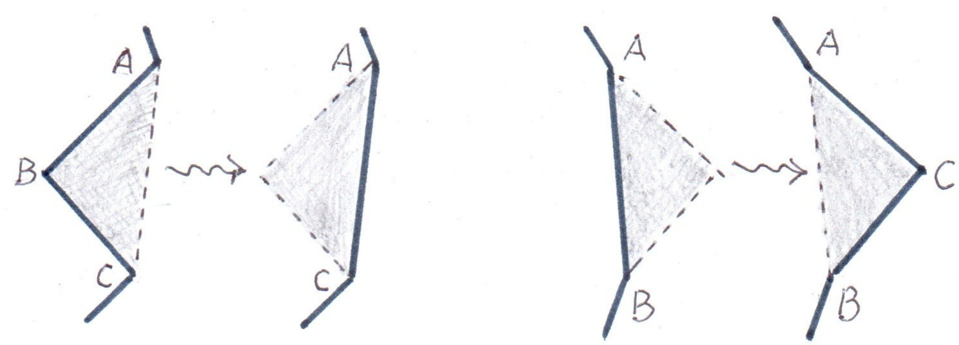


Figure 12.1. Δ-moves

An example of a sequence of Δ-moves changing the shape of a knot are shown in Figure 12.2. The figure clarifies the idea that two knots are isotopic if their practical models (made from rope) can be given the same shape by appropriately moving the ropes.

Isotopy is obviously an equivalence relation, and the word "knot" is often used in the sense of "isotopy class of knots", we often say that two closed broken lines are "the same knot" or "have the same knot type" if they are isotopic. (And similarly for links.)

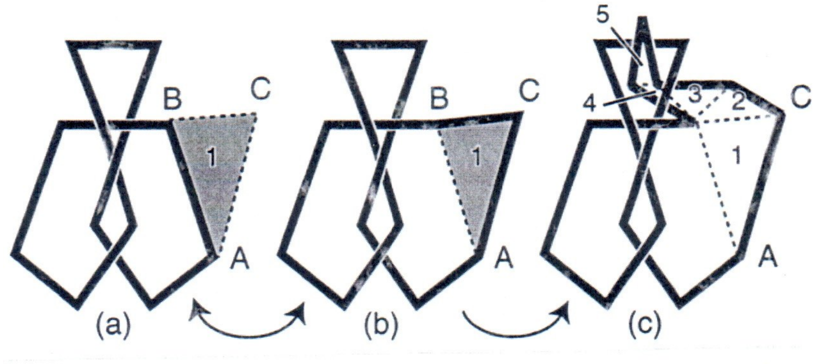


FIGURE 12.2. A sequence of Δ-moves

A knot is called *trivial* (or said to be the *unknot*) if it is isotopic to a regular polygon. Examples of famous knots and links are shown in Figure 12.2. In the figure the knots are presented in the form of *knot diagrams*, i.e., projections of the knot on the plane in general position, with underpass-overpass information at each double point (it shows which one of the branches lies above the other).

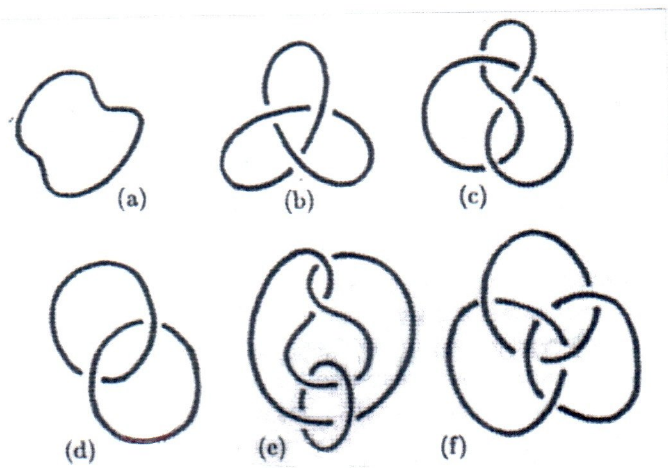


FIGURE 12.3. (a) unknot; (b) right trefoil; (c) eight knot; (d) Hopf link; (e) Whitehead link; (f) Borromeo rings.

There are several equivalent definitions of knot, link, and isotopy (e.g. smooth knots or PL-knots). The definition given above is the most elementary one. Another elementary definition of knot consists in "putting each knot in a box", i.e., defining a knot as a broken line inside a cube joining the centers of opposite faces of the cube, with isotopy being a sequence of Δ-moves performed inside the cube and not moving the endpoints of the broken line. It is easy to show that there is natural bijection between isotopy classes of boxed knots and knot types as defined above.

12.2. The arithmetic of knots

We define the *composition* (also called *connected sum*) of two (boxed) knots K_1, K_2 as the knot $K_1 \# K_2$ obtained by fitting the boxes together as shown in Figure 12.4). Under the composition operation *knots form a semigroup* denoted by $\mathcal{K}$.

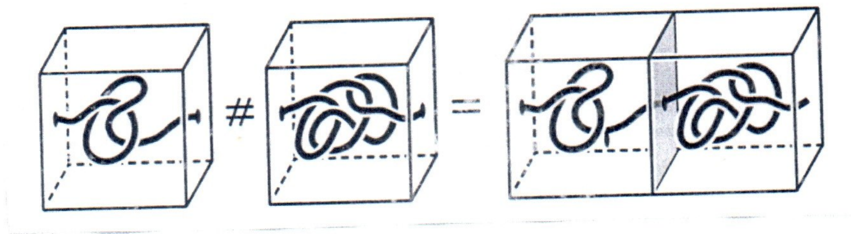


FIGURE 12.4. Composition of two knots

A knot K is called *prime* if it cannot be presented as the sum of two nontrivial knots, i.e., $K = K_1 \# K_2$ implies that either K_1 or K_2 is the unknot.

Theorem 12.2. *The semigroup $\mathcal{K}$ is commutative, it has no inverse elements* (*i.e.,* $K_1 \# K_2 = O$*, where O denotes the unknot, implies that both K_1 and K_2 are trivial*) *and each non trivial knot possesses a unique* (*up to order*) *decomposition into prime knots.*

We omit the (fairly difficult) proof of this lovely theorem, but show the isotopy demonstrating commutativity (Figure 12.5.)

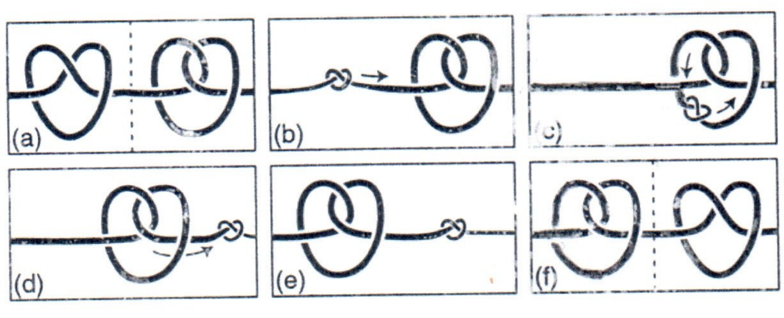


FIGURE 12.4. Commutativity of the connected sum of knots

12.3. The combinatorics of knots: Reidemeister moves

The knot classification problem is a very difficult three-dimensional geometric problem (for a detailed formulation, see Section 12.4 below), but it has been reduced to a combinatorial two-dimensional problem by Reidemeister. This reduction was done by means of certain modifications of knot diagrams called *Reidemeister moves*; they are shown in Figure 12.6. The first move, Ω_1, is the removal (addition) of a small loop, the second one, Ω_2, is the removal (addition) of an overlap, and the third, Ω_3, is the passage of a branch of the knot over a crossing point.

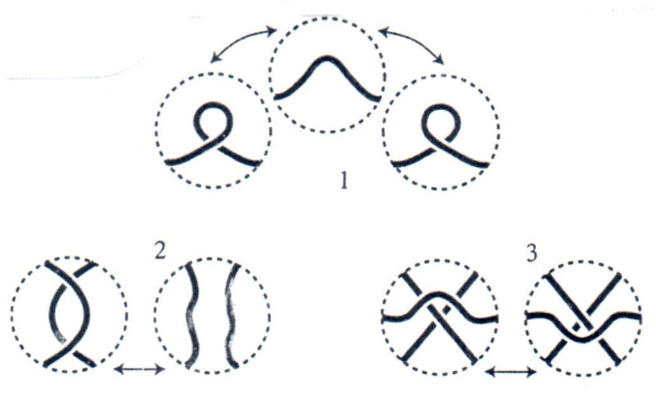


Figure 12.6. Reidemeister moves

Theorem 12.2. *If two knot (link) diagrams define isotopic knots (links), then one can be taken to the other by a finite sequence of Reidemeister moves.*

We omit the proof, which can be obtained by a general position argument.

The Reidemeister theorem did not lead to a simple solution of the knot classification problem, but turned out to be extremely useful in the construction of various knot invariants.

12.4. The Alexander-Conway polynomial

The Alexander-Conway polynomial is an invariant of oriented knots and links; it can be introduced by means of the Conway axioms: we are given a rule that to each oriented diagram of a knot or link L assigns a polynomial $\nabla_L(x)$ and satisfies the following axioms:

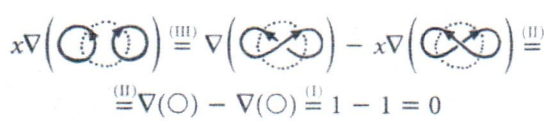
I. [Invariance] $\nabla_L(x)$ *is an isotopy invariant.*

II. [Normalization] *For the unknot* O, $\nabla_O(x) = 1$.

III. [Skein relation] *the following equality holds*

for any three link diagrams that are identical everywhere except inside the dotted circles, where they are as shown in the figure.

We will not prove that the Alexander-Conway polynomial exists and is well defined by these axioms, we only present an example of its calculation.



12.5. About the classification of knots

The solution of the *knot classification problem* is an algorithm that determines whether or not two knot diagrams define the same knot. The existence of such an algorithm was proved by S.V. Matveev a few years ago, but the algorithm is too complicated to be implemented in a computer. However, prime knots with a small (≥ 16) number of crossings have been classified (by means of invariants more powerful than the Alexander-Conway polynomial) and are tabulated in *knot tables.* A small knot table is given below.

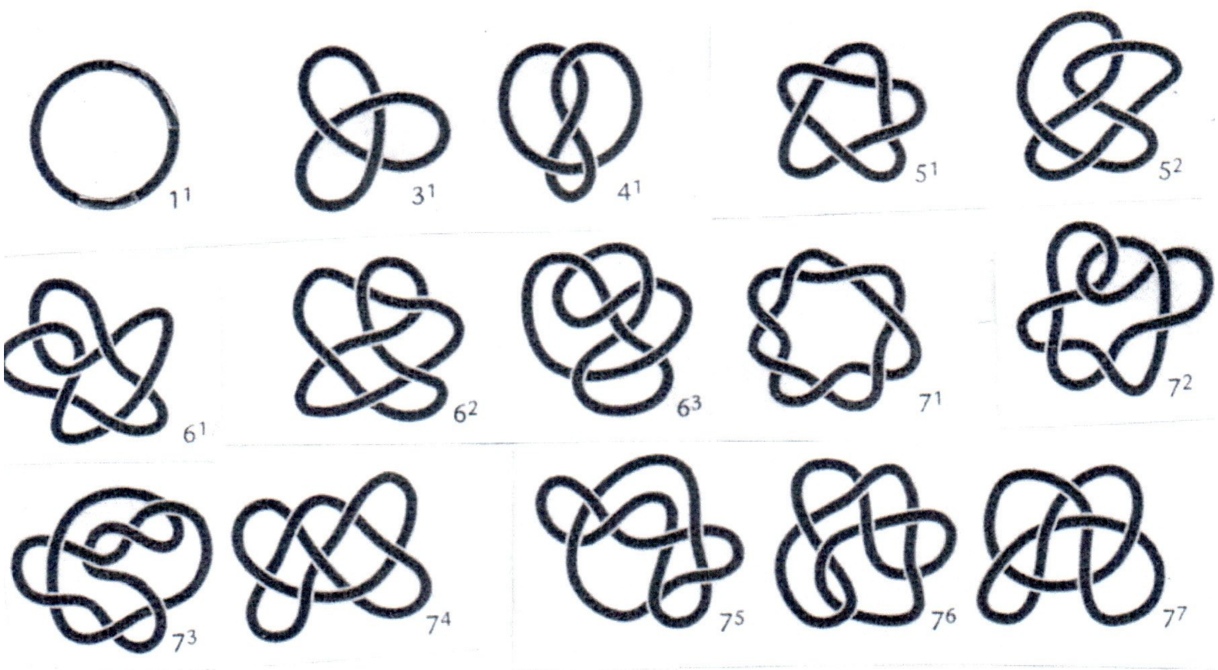
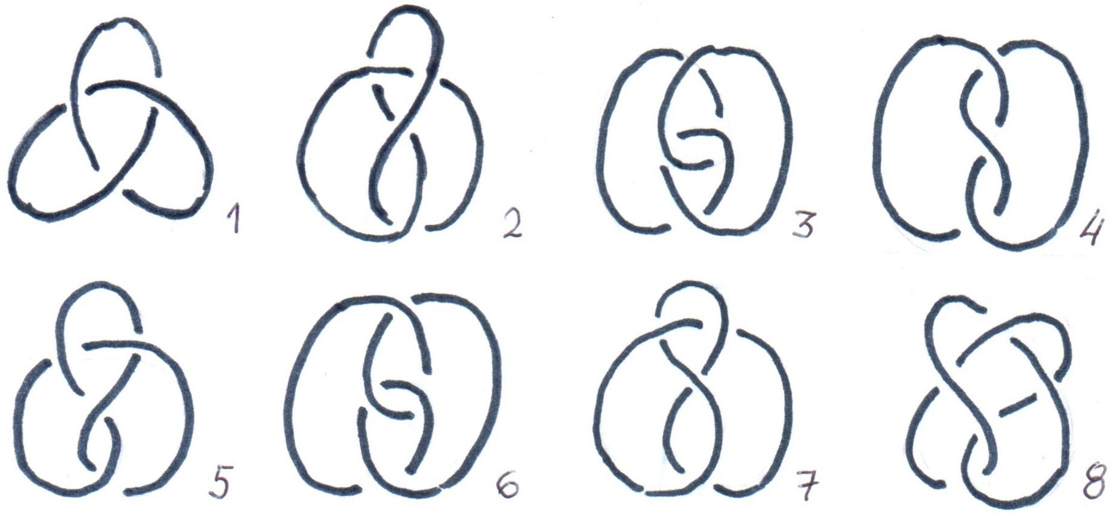


FIGURE 12.6. Table of prime knots with 7 crossings or less

The search for efficient algorithms that classify knots is continuing today, one of the approaches involves the minimization of knot energy functionals.

12.6. Exercises

12.1. Which of the knots in the picture are trivial, are trefoils, are eight knots?



12.2. Compute the Conway polynomials of the two Hopf links.

12.3. Compute the Conway polynomials of the two (right and left) trefoils.

12.4. Compute the Conway polynomials of the eight knot.