Lecture 4

EXAMPLES OF SURFACES

In this lecture, we will study several important examples of surfaces (closed surfaces, as well as surfaces with holes) presented in different ways. We will prove that the different presentations of the same surface are indeed homeomorphic and specify their simplicial and cell space structure.

4.1. The Disc $\mathbb{D}^2$

The standard two-dimensional *disk* (or 2-*disk*) is defined as

$$\mathcal{D}^2 := \{x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Other presentation of the 2-disk (all homeomorphic to $\mathcal{D}^2$) are: the sphere with one hole (SH), the square (Sq), the lateral surface of the cone (LC), the ellipse, the rectangle, the triangle, the hexagon, etc. (see Figure. 4.1).

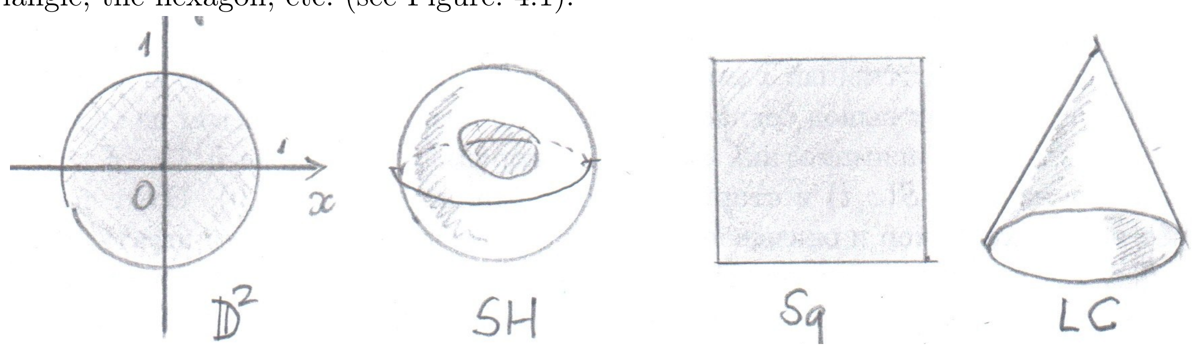

FIGURE 4.1. Different presentations of the disk

The simplest cell space structure of the 2-disk consists of one 0-cell, one 1-cell, and one 2-cell, but of course other cell space structures are possible.

It is easy to prove that the different presentations of the disk listed above are homeomorphic. For example, a homeomorphism of $\mathcal{D}^2$ onto the square Sq is obtained by centrally projecting concentric circles filling $\mathcal{D}^2$ onto the corresponding circumscribed concentric square boundaries (Figure 4.2). More precisely, we define $h : \mathcal{D}^2 \to \mathrm{Sq}$ as follows: let a point $P \in \mathcal{D}^2$ be given; denote by C_P the circle centered at the center O of the disk and passing through P; denote by D_P the boundary of the square with sides parallel to the sides of Sq circumscribed to C_P; then $h(P)$ is defined as the intersection of the ray $[OP\rangle$ with D_P. It is easy to see that h is a homeomorphism, so that the disk $\mathcal{D}^2$ and the square Sq are indeed homeomorphic.

Describing the other homeomorphisms of $\mathcal{D}^2$ (onto the sphere with one round hole (SH), the lateral surface of the cone (LC), the ellipse, the rectangle) is the object of Exercise 4.1.

2.2. The Sphere $\mathcal{S}^2$

The two-dimensional *sphere* (or 2-*sphere*) is defined as $\mathcal{S}^2 := \{x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Other presentations of the 2-sphere (all homeomorphic to $\mathcal{S}^2$) include: the boundary of the cube or the tetrahedron, the disk with boundary identified to one point $\mathcal{D}^2/\partial\mathcal{D}^2$, the suspension over the circle $\Sigma(\mathcal{S}^2)$, the join of the circle and the 0-sphere (i.e., a pair of points) $\mathcal{S}^2 * \mathcal{S}^0$, the boundary of any closed convex body, the configuration space of the 3-dimensional pendulum (the line segment in $\mathbb{R}^3$ with one extremity fixed), etc.

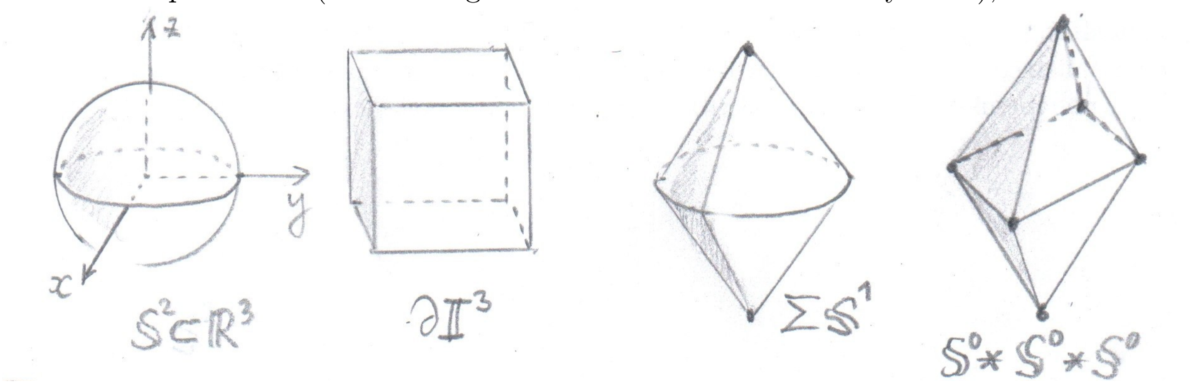

FIGURE 4.2. Different presentations of the sphere

The simplest cell space structure of the 2-sphere consists of one 0-cell and one 2-cell.

Homeomorphisms between the various presentation of $\mathcal{S}^2$ listed above are easy to construct (central projection is the main instrument here; see Exercise 4.1).

4.3. The Möbius Strip Mb

The Möbius band (or Möbius strip) Mb is usually modeled by a long rectangular strip of paper with the two short sides identified ("glued together") after a half twist (Figure 4.4). Formally it can be defined as the square with two opposite sides identified Sq/$\sim$ via the central symmetry $\sim$. A beautiful embedding of the Möbius strip Mb $\hookrightarrow \mathbb{R}^3$ can be observed as a trefoil knot spanned by a soap film; the same embedded surface can be obtained by giving a long strip of paper three half-twists and then identifying the short sides. An even more complicated embedding of the Möbius strip in $\mathbb{R}^3$ is obtained by giving a long strip of paper a large odd number of half-twists and then identifying the short sides.

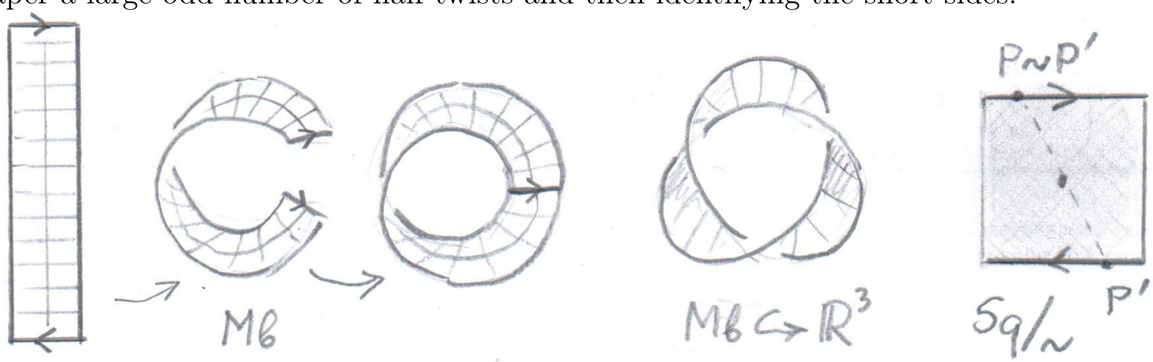

FIGURE 4.3. Different presentations of the Möbius strip

Everyone knows that the Möbius strip is "one-sided" (it cannot be painted in two colors) and is "nonorientable". (The definition of "nonorientable surface" will be given below.) If you have never done this before, try to guess what happens to the Möbius strip if you cut it along its midline. Check the validity of your guess by using scissors on a paper model.

4.4. The Torus $\mathcal{T}^2$

Topologically, the (2-dimensional) torus $\mathcal{T}^2$ is defined as the Cartesian product of two circles. Geometrically, it can be presented as the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying the equation

$$\left(x^2 + y^2 + z^2 + R^2 + r^2\right)^2 - 4R^2\left(x^2 + y^2\right) = 0.$$

The torus can also be presented as the square with opposite sides identified Sq$/\!\sim$ (the identifications are shown by the arrows in Figure 4.4), as a surface embedded (in different ways) in 3-space $\mathcal{T}^2 \hookrightarrow \mathbb{R}^3$, as a "sphere with one handle" M_1^2 (Figure 4.4 (d)), as an annulus with boundary circles (oriented in the same direction) identified, as the configuration space of the double pendulum with arms $L > l$, as the plane $\mathbb{R}^2$ modulo the periodic equivalence $(x, y) \sim (x + 1, y + 1)$, etc.

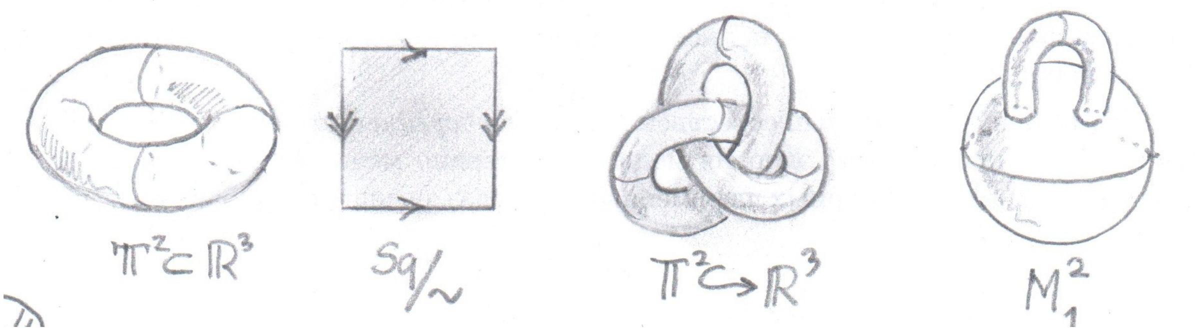

FIGURE 4.5. Different presentations of the torus

4.5. The Projective Plane $\mathbb{R}P^2$

The *projective plane* $\mathbb{R}P^2$ is defined as the set of straight lines l in $\mathbb{R}^3$ passing through the origin, with the natural topology (its base consists all open cones around all elements $l \in \mathbb{R}P^2$). The notion of straight line is naturally defined in $\mathbb{R}^2$: a "line" is a (Euclidean) plane P passing through the origin, its "points" are all the (Euclidean) lines l passing through the origin and contained in P.

Each element $l \in \mathbb{R}P^2$ may be specified by its homogeneous coordinates, i.e., the three coordinates of any point (of $\mathbb{R}^3$) on the (Euclidean) line l considered up to a common factor λ, so that $(x : y : z)$ and $(\lambda x : \lambda y : \lambda z)$, $\lambda \neq 0$, specify the same point of $\mathbb{R}P^2$.

Other presentations (Figure 4.5) of $\mathbb{R}P^2$ are: the disk $\mathcal{D}^2$ with diametrically opposed boundary points identified $\mathcal{D}^2/\!\sim$, the sphere $\mathcal{S}^2$ with all pairs of points symmetric with respect to the origin identified $\mathcal{S}^2/\mathrm{Ant}$, the sphere with a hole with a Möbius strip attached to it along the boundary $(\mathcal{S}^2 \setminus B^2) \cup_h \mathrm{Mb}$, the Möbius band with a disk glued to it along the boundary $\mathrm{Mb} \cup_f \mathcal{D}^2$, the square with centrally symmetric boundary points identified, the configuration space of a rectilinear rod rotating in $\mathbb{R}^3$ about a fixed hinge at its midpoint.

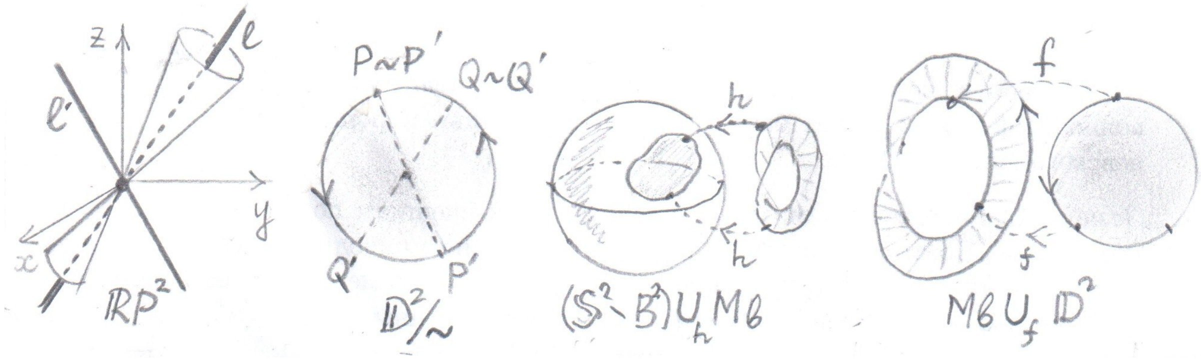

FIGURE 4.5. Different presentations of the projective plane

The proofs that all these presentations are homeomorphic are pleasant and straightforward (see Exercise 4.2).

The simplest cell space structure on $\mathbb{R}P^2$ consists of one cell in each dimension 0, 1, 2 and can be easily seen on the disk model. Note that the boundary of the 2-cell wraps around the 1-cell twice.

4.6. The Klein bottle Kl

The *Klein bottle* can be defined as the square with opposite sides identified as shown by the arrows in Figure 4.6. The Klein bottle cannot be embedded into $\mathbb{R}^3$ (see Exercise 4.12), and so we cannot draw a realistic picture of it. The Klein bottle is usually pictured as in Figure 4.6 (b), but that picture is not correct: the "surface" in the figure has a self-intersection (a little circle), so it not homeomorphic to Kl.

Here are some other presentations of the Klein bottle: two Möbius strips identified along their boundary circles $\mathrm{Mb}\cup_h\mathrm{Mb}$, a torus with a hole with boundary identified with the boundary circle of a Möbius strip $(\mathcal{T}^2 \setminus B^2)\cup_f\mathrm{Mb}$, two projective planes with holes with the boundaries of the holes identified, a projective plane with a hole whose boundary is identified with the boundary of a Möbius strip, etc.

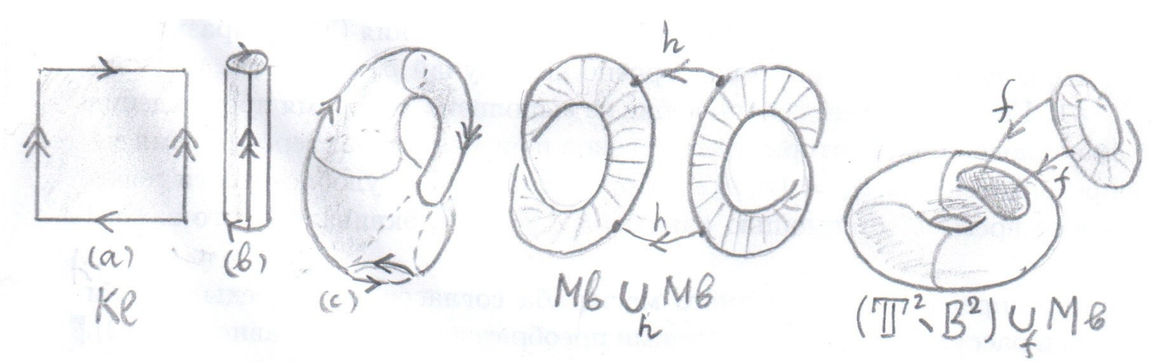

FIGURE 4.7. Different presentations of the Klein bottle

4.7. The disk with two holes ("pants")

This surface is obtained from the disk $\mathcal{D}^2$ by removing two small open disks from $\mathcal{D}^2$; it is called *pants* by topologists and denoted $\mathcal{P}$. It plays an important technical role in low-dimensional topology, in particular in the next lecture.

It is possible to construct a torus (the sphere with one handle) from two copies of pants (glue the boundaries of the four "legs" together and then close up the two "waists" by gluing disks to them). In a similar way, we can construct a sphere with $2, 3, 4, \dots$ handles.

Different ways of presenting the disk with two holes are shown in Figure 4.7 (a-d).

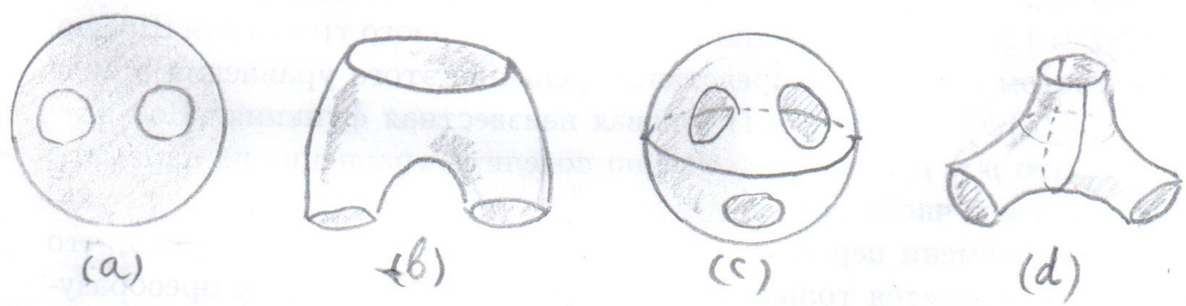

FIGURE 4.8. Different presentations of the disk with two holes

4.8. Triangulated surfaces

The surfaces (with or without holes) described above can easily be triangulated, i.e., supplied with the structure of a (two-dimensional) simplicial space. Simple examples of triangulations are shown in Figure 4.8. For triangulated surfaces the holes are usually chosen as the insides of 2-simplices, so that the boundaries of the holes will be triangles consisting of three 1-simplices. Any 1-simplex which is not on part of a boundary is the common side of two triangles (2-simplices).

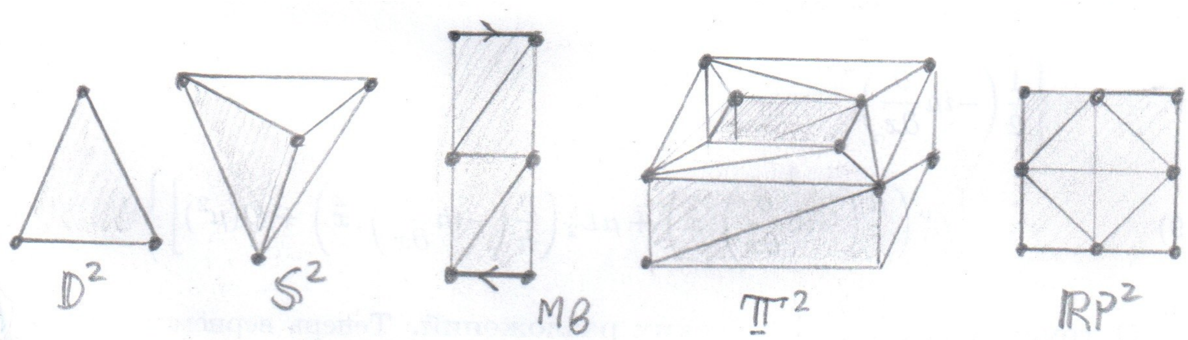

FIGURE 4.8. Triangulations of the disk, the sphere, the Möbius strip, the torus, and the projective plane (left to right).

4.9. Orientable surfaces

A triangulated surface is called orientable if all its 2-simplexes can be "oriented coherently". We do not explain what this means because, for topological surfaces, orientability can be defined in a simpler way: a surface M is called *orientable* if it does not contain a Möbius strip, and is called *nonorientable* otherwise.

It is easy to prove that *the Möbius strip, the projective plane, and the Klein bottle are nonorientable*. It is intuitively clear that *the disk, the sphere, the torus, the pants are orientable*, but this is not easy to prove. (We will come back to this question in the next lecture).

4.10. Euler characteristic

Let M be a triangulated surface, for example one of the triangulated surfaces described in Subsection 4.8. Then the *Euler characteristic* of M, denoted by $chi(M)$, is defined as

$$\boxed{\chi(M) := V - E + F},$$

where V is the number of vertices (0-simplices), E is the number of edges (1-simplices), and F is the number of faces (2-simplices) in the triangulation of the surface M.

It will be shown in the next lecture that *the Euler characteristic does not depend on the choice of triangulation, i.e.,* $\chi(M)$ *is a topological invariant:*

$$M \approx M' \implies \chi(M) = \chi(M').$$

Theorem 4.1. *The Euler characteristics of the disk, the sphere, the torus, the pants, the Möbius strip, the projective plane, and the Klein bottle are respectively equal to* $1, 2, 0, -1, 0, 1, 2$.

Since we know that $\chi(M)$ does not depend on the choice of triangulation, to prove the theorem it suffices to compute $\chi(M)$ (using its definition) for the triangulated surfaces described in Subsection 4.8.

4.11. Connected sum

Given two surfaces M_1 and M_2, their *connected sum* $M_1 \# M_2$ is obtained by removing little open disks from each and gluing them together along a homeomorphism of the little boundary circles of the removed disks. In the case when M_1 and M_2 are triangulated, it is more convenient to remove the interior of a 2-simplex in each surface and glue them together along a piecewise linear homeomorphism of the boundaries of the removed simplices.

For $M_1 \# M_2$ to be well-defined, we should prove that the connected sum does not depend on the position of the removed open disks and on the choice of the attaching homeomorphism. This can be done by a technical argument that we omit.

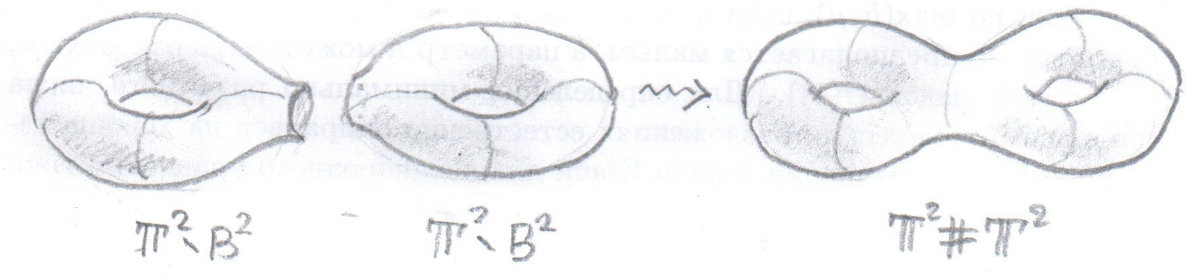

FIGURE 4.9. Connected sum of two tori

Knowing the Euler characteristics of two given surfaces M_1 and M_2, it is easy to compute the Euler characteristic of their connected sum $M_1 \# M_2$: two faces (2-simplices) have disappeared, three edges (1-simplices) have been identified with three other edges, three vertices (0-simplices) have been identified with three other vertices, so that the Euler characteristic of the sum is 2 less than the sum of the Euler characteristics of the summands. We have proved the following theorem:

Theorem 4.2. $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2.$

4.10. Exercises

4.1. Show that the surfaces in Figure 4.1, the surfaces in Figure 4.2, the surfaces in Figure 4.3, are homeomorphic.

4.2. Prove that the projective plane is (a) the Möbius strip with a disk attached; (b) the sphere S^2 with antipodal points identified; (c) the disk D^2 with diametrically opposed points identified.

4.3. Prove that the Klein bottle is (a) the double of the Möbius strip; (b) the sphere with two holes with two Möbius strips attached; (c) the connected sum of two projective planes; (d) the torus with a hole with a Möbius strip attached.

4.4. (a) Consider the topological space of straight lines in the plane. Prove that this space is homeomorphic to the Möbius band without boundary.

(b) Consider the topological space of *oriented* straight lines in the plane. Prove that this space is homeomorphic to the cylinder without boundary.

4.5. Show that a punctured tube from a bicycle tire can be turned inside out. (More precisely, this would be possible if the rubber from which the tube is made were elastic enough.)

4.6. (a) *Polygonal Schoenflies Theorem.* A closed polygonal line in the plane bounds a domain whose closure is the disk D^2.

(b) *Polygonal Annulus Theorem.* Two closed polygonal lines in the plane, one of which encloses the other, bound a domain whose closure is the annulus $S^1 \times [0, 1]$.

4.7. (a) The two surfaces with holes obtained from the same closed triangulated connected surfaces by removing two different open 2-simplices from it are homeomorphic. (b) Show that the connected sum of surfaces is well defined.

4.8. Prove that $T^2 \# \mathbb{R}P^2 \approx 3\,\mathbb{R}P^2$.

4.9. (a) Prove that $\mathrm{Kl}\#\ \mathrm{Kl}$ is homeomorphic to the Klein bottle with one handle attached. (b) Prove that $\mathbb{R}P^2 \# \mathrm{Kl}$ is homeomorphic to the projective plane with one handle attached.

4.10. Prove that if a surface M_1 is nonorientable, then for any surface M_2 the surface $M_1 \# M_2$ is nonorientable.

4.11. How many different surfaces is it possible to glue (by identifying sides) starting with (a) a square; (b) a hexagon; (c) an octagon.

4.12. Prove that the Klein bottle cannot be embedded in $\mathbb{R}^3$. (*Hint:* you can use the fact that the graph $K_{3,3}$ cannot be embedded in $\mathbb{R}^3$).