Lecture 7

Vector fields on the plane

The notion of vector field comes from mechanics and physics. Examples: the velocity field of the particles of a moving liquid in hydrodynamics, or the field of gravitational forces in Newtonian mechanics, or the field of electromagnetic induction in electrodynamics. In all these cases, a vector is given at each point of some domain in space, and this vector changes continuously as we move from point to point. In this lecture we will study a simpler model situation: vector fields on the plane (rather than in space).

In mathematics, the notion *smooth* vector field is a basic notion of differential equations (analysis) and is not a topological notion. However, in this lecture we will consider the more general (topological!) notion of *continuous* vector field and show how the notion of degree of circle maps can be used in this context, and so can be very efficiently applied to differential equations.

7.1. Trajectories and Singular Points

A *vector field* V in the plane $\mathbb{R}^2$ is an assignment of a vector to each point of the plane. In the coordinates x, y of $\mathbb{R}^2$ it may be expressed as

$$X = \alpha(x, y), \qquad Y = \beta(x, y),$$

where $\alpha\colon \mathbb{R}^2 \to \mathbb{R}$ and $\beta\colon \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions on the plane, (x, y) are the coordinates of the point p, and (X, Y) are the coordinates of the vector $V(p)$. If the functions α and β are continuous, then the vector field V is called *continuous*, and if α and β are smooth (infinitely differentiable), then V is called *smooth*. We will consider only continuous vector fields in what follows, and therefore omit the adjective "continuous".

A *singular point* p of a vector field V is a point where V vanishes: $V(p) = 0$; when V is a velocity field, such a point is often called a *rest point*, when V is a field of forces, it is called an *equilibrium point*.

A *trajectory* of the vector field V through the point $p \in \mathbb{R}^2$ is a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ passing through p and tangent at all its points to the vector field (more precisely, the vector $V(q)$ is equal to the derivative $d\gamma(t)/dt$ at each point $q \in C$). When we picture a vector field, instead of drawing numerous vectors in the plane, it is much clearer to draw its trajectories. It is a classical theorem in differential equations that *the trajectories of a smooth vector field always exist.* We do not need this theorem in the following theory, we need it only to motivate our illustrations, so will not prove it.

7.2. Generic Singular Points of Plane Vector Fields

We will now define certain types of singular points of plane vector fields. To define these points, we will not write explicit formulas for the vectors of the field, but instead describe the picture of its trajectories near the singular point and give physical examples of such singularities (see Figure 7.1).

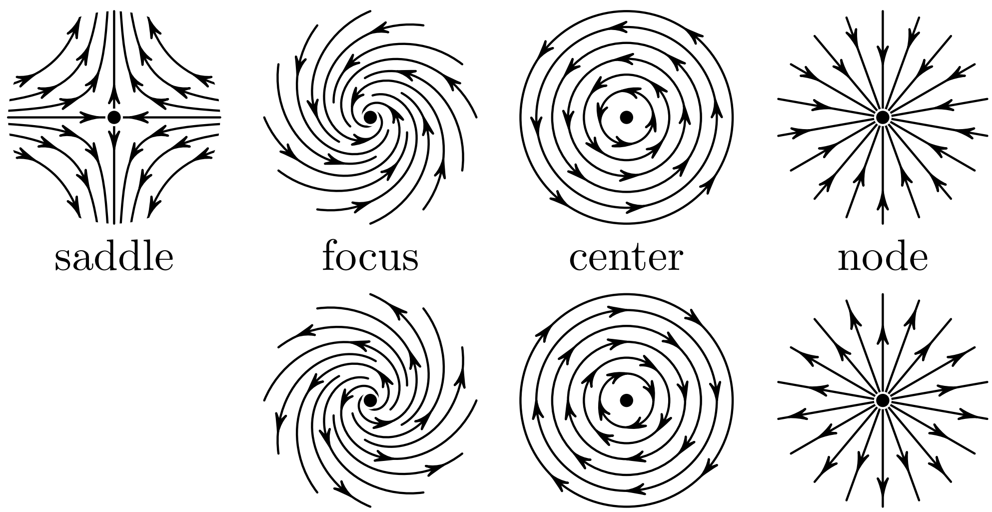

FIGURE 7.1. Singular points of vector fields

A *node* is a singular point contained in all the nearby trajectories; if all the trajectories move towards the point, the node is called *stable* and *unstable* if all the trajectories move away from the point. As an example, we can consider the gravitational force field of water droplets flowing down the surface $z = x^2 + y^2$ near the point $(0, 0, 0)$ (stable node) or down the surface $z = -x^2 - y^2$ near the same point (unstable node).

A *saddle* is a singular point containing two transversal trajectories, called *separatices*, one of which is ingoing, the other outgoing, the other trajectories behaving like a family of hyperbolas whose asymptotes are the separatrices. As an example, we can consider the gravitational force field of water droplets flowing down the surface $z = x^2 - y^2$ near the point $(0, 0, 0)$; here the separatrices are the bissectors of the coordinates axes.

A *center* is a singular point near which the trajectories behave like the family of concentric circles centered at that point; a center is called *positive* if the trajectories rotate counterclockwise and *negative* if they rotate clockwise. As an example, we can consider the velocity field obtained by rotating the plane about the origin with constant angular velocity.

A *focus* is a singular point that resembles the node, except that the trajectories, instead of behaving like the set of straight lines passing through the point, behave as a family of logarithmic spirals converging to it (*stable focus*) or diverging from it (*unstable focus*).

A singular point is called *generic* if it is of one of the three following types described above: node, saddle, focus. A vector field is called *generic* if it has a finite number of singular points all of which are generic.

Remark 7.1. Let us explain informally why the term generic is used here. Generic fields are, in fact, the "most general" ones in the sense that, first, they occur "most often" (i.e., as close as we like to any vector field there exists a generic one) and, second, they are "stable" (any vector field close enough to a generic one is also generic and has the same number of singular points). These statements are not needed in this course, so we will not make them more precise nor prove them.

Remark 7.2. It can be proved that the saddle and the center are *not* topologically equivalent to each other and not equivalent to the node or to the focus; however, the focus and the node *are* topologically equivalent; as topologists, we should not distinguish them, but we do, following the traditions of the theory of dynamical systems (where an equivalence relation stricter than homeomorphism is used). We do not use (an hence do not define) this relation.

7.3. The Index of Plane Vector Fields

Suppose a (continuous but not necessarily generic) vector field V in the plane is given. Let $\gamma(\mathbb{S}^1)$ be a closed curve in the plane (i.e., γ is an embedding (=вложение) of $\mathbb{S}^1$ into $\mathbb{R}^2$) not passing through any singular points of V; let us denote $C := \gamma(\mathbb{S}^1)$. To each vector $V(c)$, $c \in C$, let us assign the unit vector of the same direction as $V(c)$ issuing from the origin of coordinates $O \in \mathbb{R}^2$; we then obtain a map $g\colon C \to \mathbb{S}^1_1$ (where $\mathbb{S}^1_1 \subset \mathbb{R}^2$ denotes the unit circle centered at O), called the *Gauss map* corresponding to the vector field V and to the curve γ. Now we define the *index of the vector field V along the curve γ* as the degree of the circle map $(g \circ \gamma)\colon \mathbb{S}^1 \to \mathbb{S}^1$:

$$\boxed{\operatorname{ind}(\gamma, V) := \deg(g \circ \gamma).}$$

Intuitively, the index is the total number of revolutions in the positive (counterclockwise) direction that the vector field performs when we go around the curve once.

Remark 7.3. A simple way of computing $\operatorname{ind}(\gamma)$ is to fix a ray in general position issuing from O (say the half-axis Ox) and count the number of times p the endpoint of $V(c)$ passes through the ray in the positive direction and the number of times q in the negative one; then $\operatorname{ind}(\gamma) = p - q$.

Theorem 7.1. *Suppose that a simple closed curve $C = \gamma(\mathbb{S}^1)$, $\gamma : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$, does not pass through any singular points of a vector field V and bounds a domain that also does not contain any singular points of V. Then* $\operatorname{ind}(\gamma, V) = 0$.

To prove this theorem, we will need a stronger version of the Jordan Curve Theorem, known as the Schoenflies theorem, which we state as a fact without proof.

Fact (Schoenflies Theorem). *Let $C := \gamma(S^1)$ be a closed curve in the plane. Then there exists a homeomorphism h of $\mathbb{R}^2$ that takes the domain D bounded by C to the unit disk centered at the origin O.*

Proof of Theorem 7.1. Let $h : D \to \mathbb{D}^2$ be a homeomorphism given by the Schoenflies theorem of the domain D to the unit disk centered at the origin O. Consider the family of all circles $\mathbb{S}^1_r$ of radius $r \leq 1$ centered at O. Obviously,

$$. \qquad \operatorname{ind}(\gamma, V) = \operatorname{ind}(h^{-1}(\mathbb{S}^1_1), V).) \qquad (*)$$

The vector $V(h^{-1}(O))$ is nonzero, hence for a small enough r_0, all the vectors $V(s)$, $s \in h^{-1}(\mathbb{S}^1_{r_0})$, differ little in direction from $V(h^{-1}(O))$, so that we have $\operatorname{ind}(h^{-1}(\mathbb{S}^1_{r_0}), V) = 0$. But then, by continuity, $\operatorname{ind}(h^{-1}(S^1_r), V) = 0$ for all $r \leq 1$. Now the theorem follows from (*).

Now suppose that V is a smooth plane vector field and p is a singular point of V. Let C be a circle centered at p such that no other singular points are contained in the disk bounded by C. Then the *index of* V *at the singular point* p is defined as $\operatorname{ind}(p, V) := \operatorname{ind}(C, V)$. This index is well defined, i.e., it does not depend on the radius of the circle C (provided that the disk bounded by C does not contain any other singular points); this follows from the next theorem.

Theorem 7.2. *Suppose that a simple closed curve γ does not pass through any singular points of a vector field V and bounds a domain that contains exactly one singular point a_0 of V. Then* $\operatorname{ind}(\gamma, V) = \operatorname{ind}(a_0, V)$.

The proof is similar to that of Theorem 7.1 and is left as an exercise.

7.5. Exercises

7.1. On the complex plane consider the vector field $v(z) = z^n/|z|^{n-1}$ for $z \neq 0$, $v(0) = 0$. Find the index of the singular point of this field (for any integer n).

7.2. Prove that the index of the curve γ is equal to the sum of indices of the singular points that it encircles.

7.3. Suppose that two vector fields v and w are given on a closed non-self-intersecting curve in such a way that at any point X the vectors $v(X)$ and $w(X)$ do not point in exactly opposite directions. Prove that the indices of γ with respect to these vector fields are equal.

7.4.* Prove that any polynomial $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ with complex coefficients has at least one complex root.

7.5. Let us say that a vector field v is *even* if $v(x) = v(-x)$ and *odd* if $v(x) = -v(-x)$. Prove that the index of the point O for an even field is even and is odd for an odd field.

7.6. A closed self-intersecting curve divides the plane into several regions. By choosing a point O in each region, we can assign to the region the number of revolutions performed by the vector $\overrightarrow{OX}$ when the point X goes around the curve. Prove that if two regions have a common boundary, then the two numbers for the two regions differ by 1.

7.7. On the boundary circles of an annulus consider a vector field such that the vectors are tangent to the circles and the vectors at any two corresponding points of the circles have opposite directions. Extend this vector field to a vector field without singular points on the entire annulus.

7.8. Prove that a vector field given on the boundary circles of an annulus can be extended to a vector field without singular points on the entire annulus if and only if the indices of two circles are equal.

7.9. Let f be a smooth function on the plane. Prove that the index of an isolated singular point of the vector field $v = \operatorname{grad} f$

(a) can be equal to $1, 0, -1, -2, \ldots$ and

(b)* cannot be equal to the other integers.