Lecture 9

CURVES IN THE PLANE

In this lecture we study curves and points lying in the plane $\mathbb{R}^2$ and introduce two important invariants: the Whitney index of a curve $w(\gamma)$ and the degree of a point with respect to a curve $\deg(p, \gamma)$. The Whitney index will allow us to classify curves immersed in the plane up to regular homotopy and the degree of a point with respect to a curve will help us prove the so-called "Fundamental Theorem of Algebra".

9.1. Regular Curves and Regular Homotopy

A closed curve $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ is called *regular* if it has a continuously changing nonzero tangent vector at each point; this means that for any $s \in \mathbb{S}^1$ there exists a neighborhood $U \subset \mathbb{S}^1$, $s \in U$, such that the restriction $f|_U$ defines the graph of a continuously differentiable function in some coordinate system in $\mathbb{R}^2$ and this graph has a nonzero tangent vector at the point $f(s)$. Note that a regular curve can have self-intersection points and even "overlaps", i.e., its image $f(\mathbb{S}^1)$ may contain intervals that are the image of disjoint intervals of $\mathbb{S}^1$, $f(U) = f(V)$, $U \cap V = \emptyset$.

A *regular homotopy* of a curve $f\colon \mathbb{S}^1 \to \mathbb{S}^2$ is a homotopy of this curve (i.e., a map $F\colon \mathbb{S}^1 \times [0,1] \to \mathbb{R}^2$ satisfying $F(s,0) = f(s)$ for all $s \in \mathbb{S}^1$) that determines a regular curve for each $t \in [0,1]$ (i.e., the curve $F(s, t_0)$ is regular for any fixed $t_0 \in [0,1]$). Note that the "disappearance of a little loop", which can occur in a homotopy (see Fig. 9.1), is impossible in a *regular* homotopy (why?).

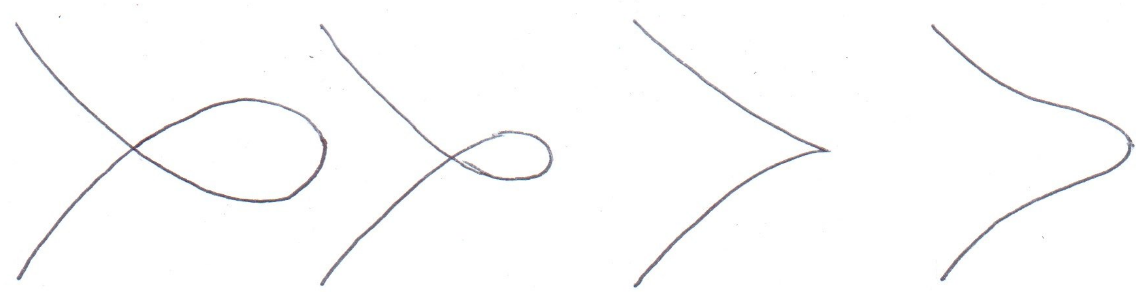

FIGURE 9.1. Disappearance of a little loop

9.2. Immersed Curves and Regular Homotopy

An immersed curve is a regular curve which is generic in the sense that its singular points cannot be destroyed by arbitrarily small changes. The exact definition is the following. A regular curve $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ is said to be an *immersion* if f is not a bijection at only a finite number of points d_j, and these points are *transversal double points*, i.e., their preimages are pairs of points and the two tangent vectors at each d_j are linearly independent.

Our aim is to classify immersed curves in the plane up to regular homotopy. This will be done by using an invariant defined in the next section.

9.3. The Whitney Index

The *Whitney index* (or *winding number*) $w(f)$ of a regular curve $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ (not necessarily immersed) is defined as the degree of the Gauss map $\gamma_{df}\colon \mathbb{S}^1 \to \mathbb{S}^1$ determined by the tangent vector to the curve; this means that γ_{df} is obtained by parallel translation of the mobile tangent vector $df(\varphi)$ to the origin and normalizing it, and then letting φ vary from 0 to 2π.

There is a simple practical method for computing $w(f)$ for an immersed curve f: we consider all the horizontal tangent vectors to f and assume that there is a finite number of them, then we count the number of these vectors of different types and combine these numbers in the appropriate way. For the details, see the exercise class.

Clearly, *the Whitney index $w(f)$ is an invariant of regular homotopy* (because it is continuous and integer-valued).

9.4. Classification of Immersed Curves

In our classification we will ignore orientation, i.e., will not distinguish a curve f from the curve $f \circ \mathrm{sym}$, where sym is the symmetry of $\mathbb{S}^1$ with respect to a diameter. This classification is given by the following theorem.

Theorem 9.1. [H. Whitney, 1928.] *Any immersed curve (up to orientation) is regularly homotopic to exactly one of the following curves: the "figure eight curve", the circle, the circle with one small loop inside it, the circle with two small loops inside it, ..., the circle with n small loops inside it, and so on.*

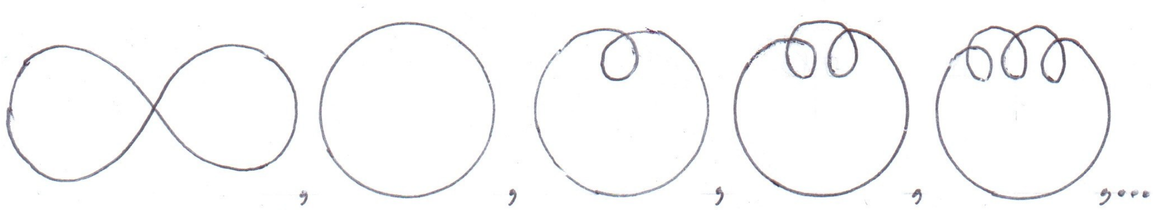

FIGURE 9.2. Classification of immersed curves

Proof. As usual for classification theorems, the proof is in two parts – one geometric, the other algebraic. In the geometric part, we construct a regular homotopy taking an arbitrary immersed curve to one of the curves listed in the theorem; we sketch this construction below (the details will be done in the exercise class). The second part consists in showing that the curves in the list are pairwise nonhomotopic; this is done by computing their Whitney indices; it turns out that they are all different (Exercise 9.10).

Let γ be the given immersed curve. We define a *simple loop* ω as a part of γ that starts and ends at a double point of γ and has no self-intersections (however, it can intersect other parts of γ).

First we prove that any immersed curve with self-intersections has a simple loop (Exercise 9.7).

Next we show that there is a homotopy after which all the simple loops do not intersect other parts of γ (Exercise 9.8)..

Finally we use the homotopies shown in Figure 9.2 (Exercises 9.3 and 9.4) to conclude the proof of the theorem (Exercise 9.5).

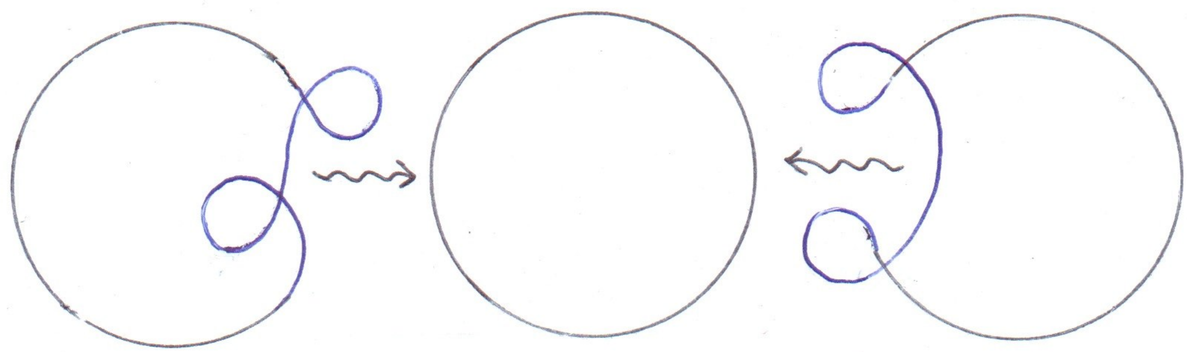

FIGURE 9.3. Two useful homotopies

Theorem 9.2. [The Whitney Theorem for the Sphere.] *Any immersed curve in the sphere is regularly homotopic to the circle or to the "figure eight curve".*

Note that here we do not classify "up to orientation" as in the previous theorem, but the classification "up to orientation" will be the same (why?). Concerning the proof, see the exercise class (Exercise 9.6).

9.5. Degree of a Point with Respect to a Curve

Consider an immersed curve $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ and a point $p \in \mathbb{R}^2$ in its complement, $p \notin f(\mathbb{S}^1)$. Let $\varphi \in [0, 2\pi)$ be the angular parameter on $\mathbb{S}^1$ and V_φ be the vector joining the points p and $f(\varphi)$. As φ varies from 0 to 2π, the unit vector $V_\varphi/|V_\varphi|$ moves along the unit circle S_0 centered at p, defining a map $\gamma_f\colon S_0 \to S_0$. The *degree of the point p with respect to the curve f* is defined as the degree of the circle map γ_f, i.e., $\deg(p, f) := \deg(\gamma_f)$.

It is easy to prove that $\deg(p, f)$ does not change when p varies inside a connected component of $\mathbb{R}^2 \setminus\ f(S^1)$ (Exercise 9.7). If the point p is "far from" $f(S^1)$ (i.e., in the connected component of $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$ with noncompact closure), then $\deg(p, f) = 0$ (Exercise 9.8).

Remark 9.2. There is a convenient method for computing the degree of any point p in the case when the curve f is immersed: join p by a (nonclosed) curve α in general position to a far away point a and move from a to p along that curve, adding one to the degree when you cross $f(\mathbb{S}^1)$ in the positive direction (i.e., so that the tangent vector looks to the right) and subtracting one when you cross it in the negative direction. The proof of the fact that you will always (independently of the choice of α) obtain $\deg(p, f)$ when your reach p is the object of Exercise 9.9.

9.6. The "Fundamental Theorem of Algebra"

The Fundamental Theorem of Algebra says that *any polynomial*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0, \quad n > 0,$$

has at least one (*possibly complex*) *root*; here the coefficients a_i may be real or complex. We will prove this theorem assuming that $a_n = 1$ and $a_0 \neq 0$; this does not restrict generality (why?).

Consider the curve $f_n \colon \mathbb{S}^1 \to \mathbb{R}^2$ given by the formula $e^{i\varphi} \mapsto R_0^n e^{in\varphi}$, where R_0 is a (large) positive number that will be fixed later. Further, consider the family of curves $f_{p,R} \colon \mathbb{S}^1 \to \mathbb{R}^2$ given by the formula

$$e^{i\varphi} \mapsto p(Re^{i\varphi}), \quad \text{where} \quad R \leq R_0.$$

We can assume that the origin O does not belong to $f_{p,R_0}(S^1)$ (otherwise the theorem is proved).

Lemma 9.1. *If R_0 is large enough, then* $\deg(O, f_{p,R_0}) = \deg(O, f_n) = n$.

Before proving the lemma, let us show that it implies the theorem.

By the lemma, $\deg(O, f_{p,R_0}) = n$. Let us continuously decrease R from R_0 to 0. If for some value of R the curve $f_{p,R}(\mathbb{S}^1)$ passes through the origin, the theorem is proved. So we can assume that $\deg(O, f_{p,R})$ changes continuously as $R \to 0$; but since the degree is an integer, it remains constant and equal to n. However, if R is small enough, the curve $f_{p,R}(\mathbb{S}^1)$ lies in a small neighborhood of a_0; but for such an R we have $\deg(O, f_{p,R}) = 0$. This is a contradiction, because $n \geq 1$.

It remains to prove the lemma. The equality $\deg(O, f_n) = n$ is obvious. To prove the other equality, it suffices to show that for any φ the difference Δ between the vectors $V_p(\varphi)$ and $V_n(\varphi)$ that join the origin O with the points $f_p(R_0 e^{i\varphi})$ and $f_n(R_0 e^{i\varphi})$, respectively, is small in absolute value (as compared to $R_0^n = |V_p(\varphi)|$) if R_0 is large enough. Indeed, by the definition of degree, if the mobile vector is replaced by another mobile vector whose direction always differs from the direction of the first one by less than $\pi/2$, the degree will be the same for the two vectors.

Clearly, $|\Delta| = |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|$. Let us estimate this number, putting $z = R_0 e^{\varphi}$ (we assume that $R_0 > 1$) and $A = \max\{a_{n-1}, a_{n-2}, \ldots, a_0\}$. We have

$$|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \leq |A(R_0^{n-1} + R_0^{n-2} + \cdots + 1)| \leq A \cdot n \cdot R_0^{n-1}.$$

Now if we put $R_0 := K \cdot A$, where K is a large positive number, we will obtain

$$|\Delta| \leq nA(KA)^{n-1} = nK^{n-1}A^n.$$

Let us compare this quantity to R_0^n; the latter equals $R_0^n = K^n A^n$, so for K large enough the ratio $|\Delta|/R_0^n$ is as small as we wish. This proves the lemma and concludes the proof of the theorem.

9.7. Exercises

9.1. Prove that any immersed curve with self-intersections has at least one simple loop.

9.2. Prove that for any simple loop ω of an immersed curve γ there exists a regular homotopy which changes only ω and replaces ω by a new simple loop that does not intersect other parts of γ.

9.3. Prove that the immersed curve shown on the left in Fig. 9.2 is regularly homotopic to the circle.

9.4. Prove that the immersed curve shown on the right in Fig. 9.2 is regularly homotopic to the circle.

9.5. Using the results of Exercises 9.1-9.4, prove the Whitney Theorem.

9.6. Prove the Whitney Theorem for the sphere.

9.7. Prove that $\deg(p, f)$ does not change when p varies inside a connected component of $\mathbb{R}^2 \setminus f(S^1)$.

9.8. Prove that if the point p is "far from" $f(S^1)$ (i.e., in the connected component of $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$ with noncompact closure), then $\deg(p, f) = 0$.

9.9. Prove that the algorithm described in Remark 9.1 finds an integer d (which is independent of the choice of the curve α) and this integer is the degree: $\deg(p, f) = d$.

9.10. Compute the Whitney index of the curves shown on Figure 9.1.