1. Let $\hat{f}(\xi)$ denote the Fourier transform of a (Schwarz) function $f(x)$ on $\mathbb{R}^n$. Put for any $a = a(x, \xi) \in C(\mathbb{R}^{2n})$ (with suitable decay conditions)

$$(Op(a)f)(x) = \int_{\mathbb{R}^n} a(x, \xi)\hat{f}(\xi)e^{2i\pi x\cdot\xi}d\xi.$$

   Find $\hat{Q}_i = Op(x_i)$, $\hat{P}_i = Op(\xi_i)$ and $Op(x \cdot \xi)$.

2. Find the eigenvalues and eigenfunctions of the momentum operator $\hat{P} = \sum_j \hat{P}_j$ in standard representation (see previous problem). Same for $\hat{Q} = \sum_j \hat{Q}_j$.

3. In previous notation, let $n = 3$. Find the expression for the angular momentum $\hat{L} = \hat{Q} \times \hat{P} = \sum_j \hat{Q}_j\hat{P}_j$ and its square in Euclidean spherical coordinates.

4. *Legendre's polynomials* $P_l(x)$ are determined by the formula

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_l P_l(x)t^l.$$

   Prove the following properties of $P_l(x)$:

   (a) $(l + 1)P_{l+1}(x) - (2l + 1)xP_l(x) + lP_{l-1}(x) = 0$;

   (b) $P_l'(x) - 2xP_{l-1}'(x) + P_{l-2}'(x) = P_{l-1}'(x)$;

   (c) $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l$ (Rodriguez recurrence formula);

   (d)

$$(1 - x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d}{dx}P_n(x) + n(n + 1)P_n(x) = 0$$

   (Legendre differential equation).

5. Let $P_n^m(x) = (1 - x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$ be the *associated Legendre polynomial*. Prove that it verifies the associated Legendre differential equation:

$$(1 - x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d}{dx}P_n(x) + \left(n(n + 1) - \frac{m^2}{1 - x^2}\right)P_n(x) = 0.$$

6. * Use previous results and the information to compute eigenvalues and eigenfunctions of the square of the angular momentum $\hat{L}^2$ and of the z-component of the angular momentum operator $\hat{L}_z$.

7. Let $\mathcal{H}$ be a Hilbert space, L any (real) vector space, L^* its dual and $U : L \to \mathcal{U}(\mathcal{H}), V : L^* \to \mathcal{U}(\mathcal{H})$ be two representations of the Abelian groups in $\mathcal{H}$ such that

$$V(f)U(x) = e^{if(x)}U(x)V(f).$$

   This is called *a Weyl representation of L*.

(a) Check the formula

$$W(z+z') = e^{-\frac{1}{2}\omega(z,z')} W(z)W(z')$$

for $z = (x, f), z' = (x', f') \in V = L \oplus L^*$ with standard symplectic structure, where $W(z) = e^{i\frac{1}{2}f(x)} U(x)V(f)$.

(b) Let $\hat{P}(x)$ be the infinitezimal generator of $t \mapsto U(tx)$, $x \in L$; let $\hat{Q}(x)$ be the infinitezimal generator of $t \mapsto V(tF(x))$, where $F : L \to L^*$ is an isomorphism, determined by a choice of basis. Show that the operators $a_i = \frac{1}{\sqrt{2}}(\hat{Q}(e_i) + i\hat{P}(e_i))$, $a_i^\dagger = \frac{1}{\sqrt{2}}(\hat{Q}(e_i) + i\hat{P}(e_i))$ where $e_1, \ldots, e_n$ is the basis used before, verify the relations:

$$[a_j, a_k] = 0, \quad [a_j^\dagger, a_k^\dagger] = 0, \quad [a_j, a_k^\dagger] = \delta_{jk}.$$