1. The "Poisson-Chern-Simons class":

   (a) Let θ be a connection form on a vector bundle, F its curvature (we regard them as locally determined matrix-valued forms on the base). Prove, that $cs_3 = Tr(F \wedge \theta + \frac{2}{3}\theta \wedge \theta \wedge \theta)$ verifies the equation $d(cs_3) = Tr(F \wedge F)$.

   (b) Let ∇ be a torsion-free connection on M; consider for each $X \in Vect(M)$ the linear map $A^\nabla(X) : TM \to TM$, $Y \overset{A^\nabla(X)}{\mapsto} \nabla_Y X$. Also let $R^\nabla : \wedge^2 Vect(M) \to End(TM)$ be the map, determined by the curvature of ∇. Prove that the cochain $T'_\theta \in C^3_{CE}(Vect(M), C^\infty(M)$, given by

   $$T'_\theta(X,Y,Z) = Asym(Tr(\frac{2}{3}A^\nabla(X)A^\nabla(Y)A^\nabla(Z) + R^\nabla(X,Y)A^\nabla(Z)),$$

   is not of the form $\mu^*\alpha,\ \alpha \in \Omega^3(M)$.

   (c) Show, that if the first Pontrjagin class on M is trivial, one can find a differential 3-form α on M, such that $\delta_{CE}(T'_\theta - \mu^*\alpha) = 0$. Prove that the class of this cocycle does not depend on the choice of α.

   (d) * Prove that this class T_θ is nontrivial.

2. The Vey class:

   (a) Check that the maps μ^* and μ' from the lecture (see problems 4 (b), 4 (c) below) commute with the Chevalley-Eilenberg differentials.

   (b) In the notation of previous problem, prove that the map

   $$(X,Y,Z) \mapsto \mathcal{L}_X(\nabla)(Y,Z) = [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y([X,Z]),$$

   where X, Y, Z are vector fields on M, is $C^\infty(M)$-linear in all three arguments. Thus, $\mathcal{L}_X(\nabla)$ is a differential 1-form on M with values in $End(TM)$.

   (c) Check that $\tilde{S}^3_\Gamma(X,Y) = Tr(\mathcal{L}_X(\nabla) \wedge \mathcal{L}_Y(\nabla))$ is a cocycle in $C^2_{CE}(Vect(M), \Omega^2)$.

   (d) Prove that if ∇ is symplectic, then $S^3_\Gamma = \mu'\tilde{S}^3_\Gamma$ is cocycle.

   (e) * Prove that S^3_Γ is not closed.

3. Preliminary constructions for the quantization of exact symplectic manifolds:

   (a) Prove that the existence of the field ξ on a symplectic manifold (M, ω) such that $\mathcal{L}_\xi \omega = \omega$ is equivalent to the exactness of the form ω. This type of symplectic manifolds is called *exact*. Find such ξ for the cotangent bundle of a smooth compact manifold.

   (b) Let M, ω be an exact symplectic manifold, let ξ be the vector field for which $\omega = d\iota_\xi\omega$. We use the Lie derivative on polydifferential operators to extend the action of ξ to the differentiable Chevalley complex of $C^\infty(M)$ (regarded as Lie algebra with respect to the Poisson brackets) with coefficients in differential forms on M. Let ∂ denote the Chevalley-Eilenberg differential and X_u the Hamiltonian vector field, associated function $u \in C^\infty(M)$. Prove the following identities:

i. $\mathcal{L}_\xi \partial = \partial \mathcal{L}_\xi - \partial$;

ii. $\mathcal{L}_\xi X_u = X_{\mathcal{L}_\xi u} - X_u$.

(c) In the notations of previous part, let μ^* be the map from the the Chevalley complex of vector fields on M (with coefficients in differential forms on M) into the Chevalley complex of functions on M, given by substituting the Hamiltonian fields as the arguments of cochains. Also, let μ' be the map $C^{\cdot}_{CE}(Vect(M), \Omega^2(M)) \to C^{\cdot}_{CE}(C^\infty(M), C^\infty(M))$ given by $\mu'(c) = \langle \omega, \mu * (c) \rangle$. Prove the formulas:

i. $\mathcal{L}_\xi \mu^*(C) = \mu^* \mathcal{L}_\xi C - p\mu^* C,\ C \in C^p_{CE}(Vect(M), \Omega^k(M))$;

ii. $\mathcal{L}_\xi \mu'(\Phi) = \mu' \mathcal{L}_\xi \Phi - (p+1)\mu'\Phi,\ \Phi \in C^p_{CE}(Vect(M), \Omega^2(M))$.

(d) Prove that for all ξ the map $\mathcal{L}_\xi$ commutes with the differential on the Chevalley complex of $Vect(M)$ with coefficients in $\Omega^*(M)$. Also show that $\mathcal{L}_\xi + p$ (resp. $\mathcal{L}_\xi + (p+1)$) sends cocycles of the form $\mu^* C$ (resp. $\mu'\Phi$) to coboundaries (notation as above).

(e) Prove that for all real $k \neq 2, 3$ (resp. $k \neq 3, 4$) the map $\mathcal{L}_\xi + k$ induces on $H^2_{CE}(C^\infty(M))$ (resp. on $H^3_{CE}(C^\infty(M))$) a bijection.

(f) ** Use these constructions to show that there exists a unique formal deformation of the Poisson structure on an exact symplectic manifold, such that $P_2 = rS^3_\Gamma$ in it (see papers of Lecomte and de Wilde).