Lecture 3

The Kauffman Bracket

In this lecture, we study the Kauffman bracket. In our course, it plays an important, but auxiliary role: it is needed only to define the famous Jones polynomial (this will be done in the next lecture). The Kauffman bracket, like the Conway polynomial, assigns a polynomial to each link diagram, but the links in question are assumed *nonoriented*.

The Kauffman bracket is a fundamental tool in physics, more important than the Jones polynomial (in particular in quantum field theory), although the Jones polynomial also has a significant relationship with physics (2-D statistical models).

3.1. Digression: statistical models in physics

In this section, we give a very rough idea of the notion of two-dimensional statistical model, which has no direct bearing on knot theory, but will serve, nevertheless, as the inspiration for the definition of the main protagonist of this lecture – the Kauffman bracket, which will be a kind of phony "partition function" of a 2-D "state model" determined by the given knot diagram. Let me explain what the words in quotation marks mean.

Roughly speaking, a 2-D statistical (or state) model is a system consisting of huge number of particles $\{p_i\} = P$, represented by points in the plane and joined by lines which indicate interactions between particles. The particles can be in one of two states (spins): "spin up" and "spin down" (traditionally shown by vertical vectors pointing up and down, respectively) and the *state* of the system is a picture of the particles supplied with spins and their interactions – see Fig. 3.1.

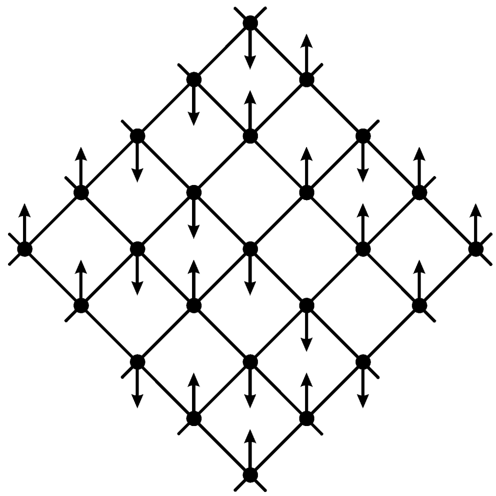


Figure 3.1. A state of a small part of a 2-D statistical model

A given model P consisting of n particles obviously has 2^n states, denoted $s \in \mathcal{S}(P)$. The main characteristic of such a model, called its *partition function*, is given by a formula such as

$$Z(P) = \sum_{s \in \mathcal{S}(P)} \exp \frac{-1}{kT} \sum_{p_i, p_j \in P} \varepsilon(s(p_i), s(p_j)),$$

where $\varepsilon(\cdot, \cdot)$ is a real-valued function expressing the "energy" of interaction between two interacting (i.e., joined by a line in the picture) particles, T is "temperature", and k is the "Boltzman constant". We shall not explain the meaning of the words in quotation marks in the above sentence – all we need to know is that the formula is the sum over all states of the product of something called interactions (the product – because of the main property of the exponential function).

3.2. The "state" of a (nonoriented) knot diagram

Suppose we consider a crossing point of a nonoriented knot (or link). Unlike crossing points of an oriented knot, among crossing points of nonoriented ones, we cannot distinguish positive and negative crossings, as we did in the Conway skein relation. However, at each crossing point, there are two vertical angles of two different kinds, that we call *A-angles* and *B-angles* (see Fig. 3.2 (a)).

A-angles are characterized by the fact fact that an observer moving along the overpass first sees the A-angle to his right, and then (after passing the crossing point), to his left. For B-angles, it's the other way around: an observer moving along the overpass first sees the B-angle to his left, and then (after passing the crossing point), to his right.

Now let us consider a (nonoriented) link diagram (such as the one in Fig. 3.2). At each of the crossing point, let us choose either an A-angle or a B-angle, and indicate the chosen angle by drawing a short "stick" in it. Such a choice is called a *state* of the link diagram (look at Fig. 3.2 (б)). If the link diagram has n crossing points, there will be 2^n possible states. In the figure, we have shaded the A-angles, and the picture acquires a chessbord coloring.

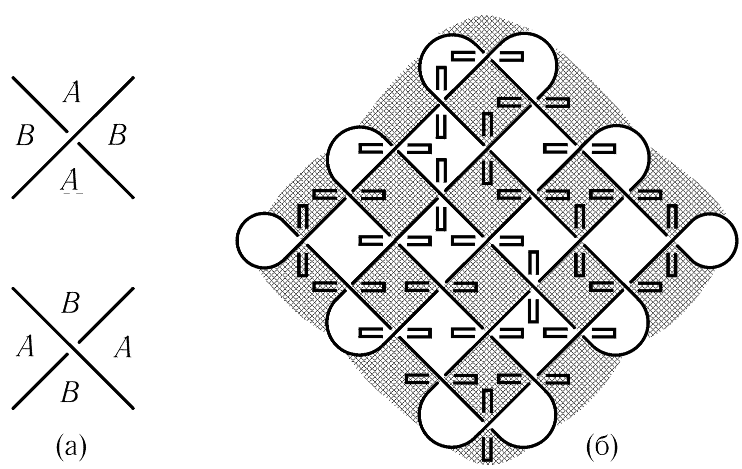


Figure 3.2. A-angles and B-angles and a state of a knot

Now let us look at a possible state of a simpler, more familiar knot, say the eight knot (Fig.3.3(a)).

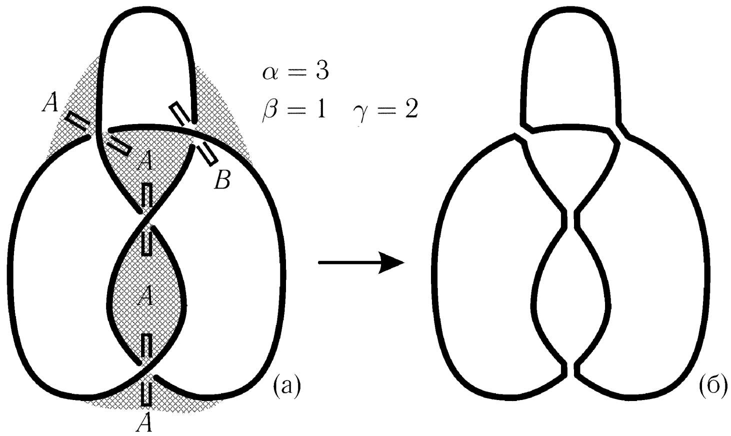


Figure 3.3. A state of the eight knot

For a given state $s \in \mathcal{S}$, let us denote the number of chosen A-angles by $\alpha(s)$, and the number of B-angles by $\beta(s)$. Obviously, $\alpha(s) = n - \beta(s)$. Now let us "smooth out" the crossing points along the sticks as shown in Fig.5.3(б). We then obtain a certain number of topological circles; this number is denoted by $\gamma(s)$. In our example, $\alpha(s) = 2$, $\beta(s) = 1$, $\gamma(s) = 2$.

3.3. Definition and properties of the Kauffman bracket

The Kauffman bracket assigns to a each link diagram K a polynomial with integer coefficients in three variables a, b, c, denoted by $\langle K \rangle$ and defined by the formula

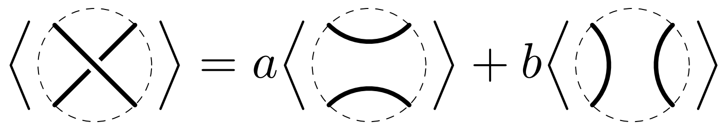
$$\langle K \rangle : = \sum_{s \in \mathcal{S}} a^{\alpha(s)} b^{\beta(s)} c^{\gamma(s)-1}.$$

Like the formula for the partition function, here we have a sum over all states of certain products. The calculation of the value of $\langle \cdot \rangle$ in our example (the eight knot) is the object of Exercise 3.1.

It easily follows from the definition that the Kauffman bracket possesses the following properties:

(I) *Normalization*: $\langle \bigcirc \rangle = 1$.

(II) *Skein relation*:



The skein relation for the Kauffman bracket differs from the Conway skein relation. It should be understood as follows: we have three nonoriented link diagrams $L^{\times}$, $L^{=}$, $L^{||}$ that are identical outside three small disks, and are as shown in the figure inside the disks; the relation says that the bracket of the diagram with a crossing point inside the disk is equal to the sum of the brackets of the two other brackets with coefficients a and b, i.e., we have

$$\langle L^{\times} \rangle = a\langle L^{=} \rangle + b\langle L^{||} \rangle.$$

(III) *Adding the unknot*: $\langle L \sqcup \bigcirc \rangle = c \cdot \langle L \rangle$.

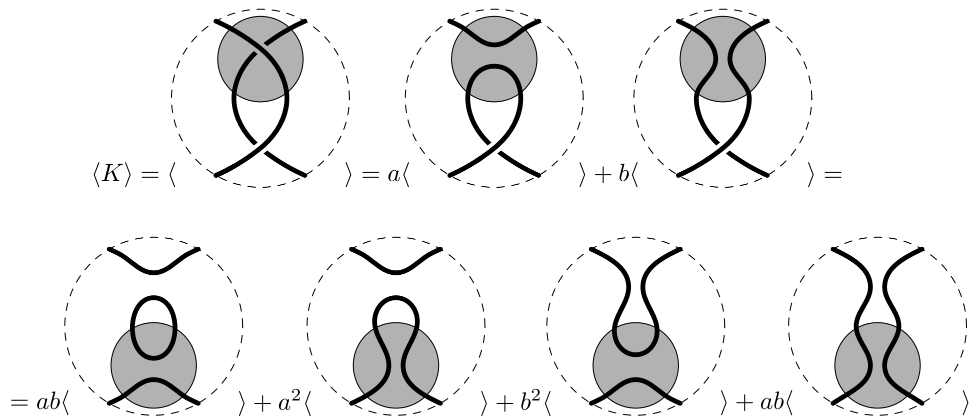
Here the left-hand side of the equality $L \sqcup \bigcirc$ is the link diagram consisting of the link L and a topological circle that does not intersect L. The equality shows that adding such a circle to a link results in its Kauffman bracket being multiplied by c.

3.5. Is the Kauffman bracket invariant?

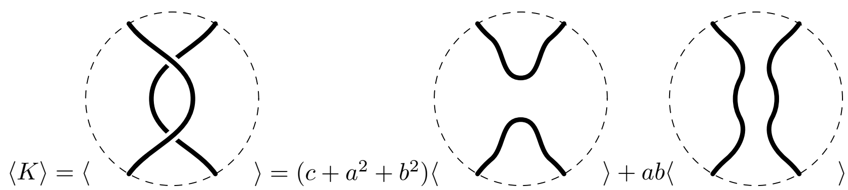
To check this, we will verify the invariance of the Kauffman bracket w.r.t. the Reidemeister moves Ω_2, Ω_3, Ω_1. If it is invariant w.r.t. all three, it will follow by the Reidemeister Lemma that $\langle \cdot \rangle$ is an isotopy invariant.

We begin with Ω_2, and consider an arbitrary link which has, inside a little disk, two arcs, one of which overpasses the other twice. Our goal is to check that its Kauffman bracket is equal to the bracket of the same link, but with two nonintersecting arcs in the little disk.

First, we apply the skein relation (II) to the upper crossing point inside the disk under consideration, then apply it to the lower crossing points of the two new disks, obtaining

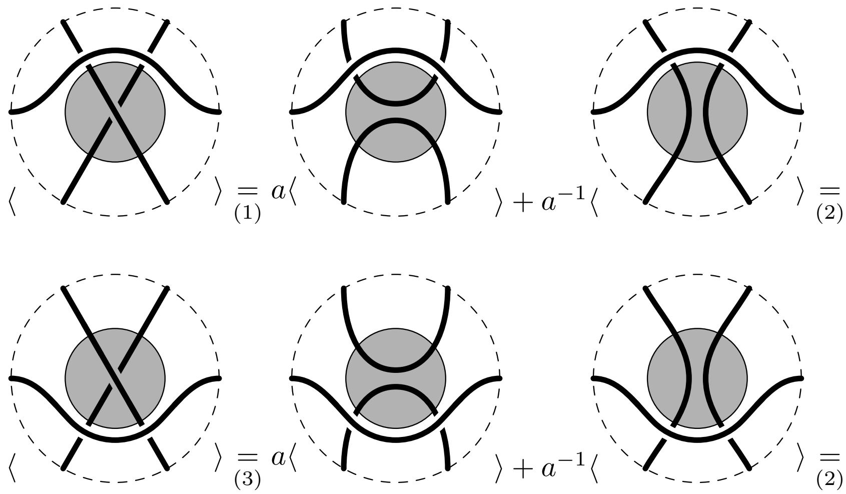


We now apply property (III) to the second summand of the last line, and after gathering like terms, we have



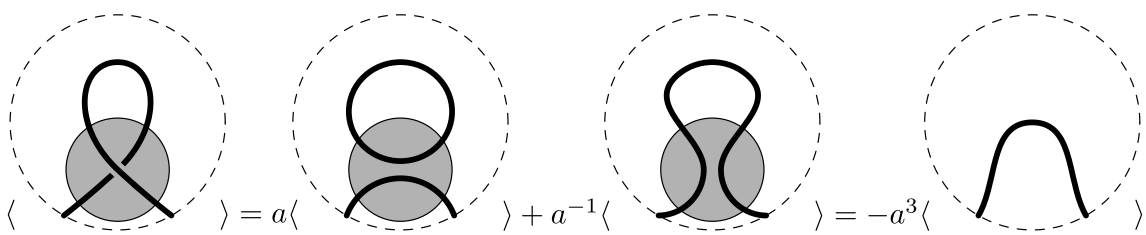
The result is not what we wanted, but if we simplify the bracket by putting $b = a^{-1}$ and $c = -a^2 - a^{-2}$, we obtain the desired result – namely the (simplified) bracket of our link with the disk containing two nonintersecting arcs. From now on, by abuse of notation, we shall denote the simplified bracket (which will be a Laurent polynomial in the variable a) by the same symbol as the old one.

The proof of the Ω_3-invarance of the (simplified) bracket appears in the figure below.

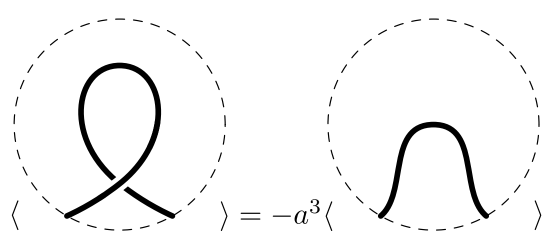


The first line of the equation should be read from left to right (we apply the skein relation again), the second line, from right to left, after having noted that the equality between the middle summands is obtained by applying Ω_2 twice (which is legal, as we have just shown).

It now only suffices to prove that our bracket is Ω_1-invariant. We have



Thus we see that *the Kauffman bracket is not* Ω_1*-invariant.* For the other little loop, we have similarly



Let us summarize our results in the following theorem.

Theorem 3.1 *The (simplified) Kauffman bracket* $\langle\,\cdot\,\rangle$ *has the following properties:*

(I) $\langle \bigcirc \rangle = 1$;

(II) $\langle L^{\times} \rangle = a\langle L^{=} \rangle + a^{-1}\langle L^{||} \rangle$;

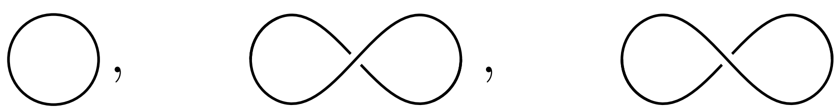
(III) $\langle L \sqcup \bigcirc \rangle = -(a^2 + a^{-2}) \cdot \langle L \rangle$;

(IV) $\langle\,\cdot\,\rangle$ *is* Ω_2*- and* Ω_3*-invariant*

(V) $\langle\,\cdot\,\rangle$ *is not* Ω_1*-invariant, the application of the* Ω_1*-move results in the multiplication of the bracket by the coefficient* $(-a)^{\pm 3}$*, where the sign depends of the type of the disappearing little loop.*

3.6. Exercises

3.1. Compute the Kauffman bracket of the following knot diagrams



3.2. Compute the Kauffman bracket of the knot diagram of the left trefoil and of the right trefoil.

3.3. Compute the Kauffman bracket of diagram of the eight knot shown in Fig. 3.3.

3.4. Compute the Kauffman bracket of the knot diagram of the knot 5_2 shown in the knot table 2.2.

3.5. Compute the Kauffman bracket of the knot diagram of the knot 5_1 shown in the knot table 2.2.

3.6. Give a detailed proof of property (II) of the Kauffman bracket (the Kauffman skein relation).

3.7. Give a detailed proof of property (III) of the Kauffman bracket.

3.8. Show that the Kauffman bracket is mutiplcative w.r.t. to the connected sum of knots.

3.9. Compute the Kauffman bracket of the granny knot. *Hint*: use the resullts of Exercises 3.2 and 3.8.