# Lecture 6

## Vassiliev Invariants

In this lecture, we begin the study of Vassiliev invariants, a.k.a. Gusarov-Vassiliev invariants or finite-type knot invariants. There is an infinite series of Vassiliev invariants, each one is a real-valued (or complex-valued, if we prefer) function $v : \mathcal{K} \to \mathbb{C}$ defined on the set of oriented knots.

Vassiliev's construction of his knot invariants is a particular case of a very general construction, which may be called the Thom–Arnold–Vassiliev discriminant method. The main idea of the method is the following. In order to study the given nice generic objects (say knots, i.e., smooth embeddings $\mathbb{S}^1 \mapsto \mathbb{R}^3$) we study, simultaneously with the nice objects, their singular analogs (singular knots, i.e., smooth maps $\mathbb{S}^1 \mapsto \mathbb{R}^3$). The set $S$ of all singular knots has a natural infinite-dimensional linear space structure, in which knots form an everywhere dense open subset $\mathcal{K}$; the set $\Delta := S \setminus \mathcal{K}$ is the discriminant of the system, it is stratified as $\Delta = \Delta_1 \cup \Delta_2 \cup \dots$, where $\Delta_1$ is the set of singular knots with one transversal self-intersection, $\Delta_2$ is the set of singular knots with two transversal self-intersections, and so on. The set $\Delta_1$ is of codimension 1 in the linear space $S$ and divides $S$ into compartments, in which the all points (knots) have the same values of any given Vassiliev invariant $v$. When we move from one compartment to an adjacent one, the value of $v$ undergoes a jump determined by the "Vassiliev skein relation". For the details, see below.

We shall introduce the Vassiliev invariants axiomatically, assuming that functions satisfying the given axioms exist, learn how to compute them in simple cases, and study their properties.

There are two different ways of proving the existence of Vassiliev invariants. The first, Vassiliev's original proof based on the Vassiliev cohomology spectral sequence, lies outside the framework of this course. The second is based on the Kontsevich integral, which we shall briefly discuss in one of the subsequent lectures.

It is conjectured that Vassiliev invariants are complete, i.e., two knots $k_1, k_2$ are not ambient isotopic if and only if there exists a Vassiliev invariant $v : \mathcal{K} \to \mathbb{C}$ such that $v(k_1) \neq v(k_2)$. It can be shown that Vassiliev invariants are stronger than the Jones polynomial: the coefficients at different powers of $q$ of the Jones polynomial of a knot can be recovered from the values of certain Vassiliev invariants of that knot.

## 6.1. Basic definitions

Denote by $\mathcal{K}$ the set of oriented knots, i.e., smooth emdeddings $k \colon \mathbb{S}^1 \to \mathbb{R}^3$. By $\Sigma$ denote the set of immersions $i \colon \mathbb{S}^1 \to \mathbb{R}^3$, i.e., smooth maps whose singularities consist of a finite number of transversal self-intersections. In particular, $\Sigma_0 = \mathcal{K}$, elements of $\Sigma_1$ have one self-intersection, those of $\Sigma_2$ have two, and so on. We have the following infinite sequence

$$\mathcal{K} = \Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_n \subset \cdots \subset \Sigma = \bigcup_{n=0}^{\infty} \Sigma_n.$$

Elements of $\Sigma_n$ for $n \geq 1$ are called *singular knots* with $n$ double points, the set $\Delta = \cup_{n \geq 1} \Sigma_n$ is the *discriminant*, the set $\Sigma_1$ is the *generic part* of the discriminant $\Delta$.

The set $S$ of all smooth maps $\mathbb{S}^1 \to \mathbb{R}^3$ has a natural infinite-dimensional linear space structure denoted by $\mathcal{C}$, the closure of $\mathcal{K}$ is the whole space $\mathcal{C}$, the closure $\overline{\Sigma_1}$ of $\Sigma_1$ divides $\mathcal{C}$ into *compartments* all points of which are knots ambient isotopic to each other, the generic part of the discriminant $\Sigma_1$ in the neighborhood of each of its points is a surface of codimension 1 in the linear space $S$. We fix an orientation on $\Sigma_1$.

**Remark 6.1.** Note that $\Sigma$ does not coincide with whole linear space $S$: for example, any smooth curve with a single triple transversal self-intersection point lies in $S$, but does not belong to $\Sigma$. The stratifications of $\Sigma$ and of $\Delta$ indicated above do not coincide with the stratifications used by Vassiliev in his original paper, but the approach developed here is based onthe same principles.

Suppose that two knots $k^+, k_-$ living in adjacent compartments are joined by a path $\alpha$ that transversally intersects the generic part $\Sigma_1$of $\Delta$ between the compartments at a point $k^0$. This point represents a singular knot with exactly one double point. Then as we move along $\alpha$, a crossing change at the point corresponding to the double point of $k^0$ occurs, a $k^+$ is transformed into $k_-$.

This is (very schematically!) illustrated in Fig.6.1, where $k^+$ is a trefoil that becomes an unknot after going through $\Sigma_1$. The knots on $\Sigma_1$ are all singular with one double point, as is is the knot $k^0$. In the figure, $\Sigma_1$ is represented as a graph in the plane, although actually it is an infinite-dimensional (nonlinear!) subspace of codimension one in infinite-dimensional linear space. The selfintersections of $\Sigma_1$ are shown as points (vertices of the graph), although actually they are also nonlinear infinite-dimensional subsets $\Sigma_2$(of codimension two), its points being singular knots with two double points. The "deeper parts" of the discriminant ($\Sigma_k$, $k \geq 3$) are not represented by anything in the figure, which is very schematical and gives no idea of the complexity of the overall picture.

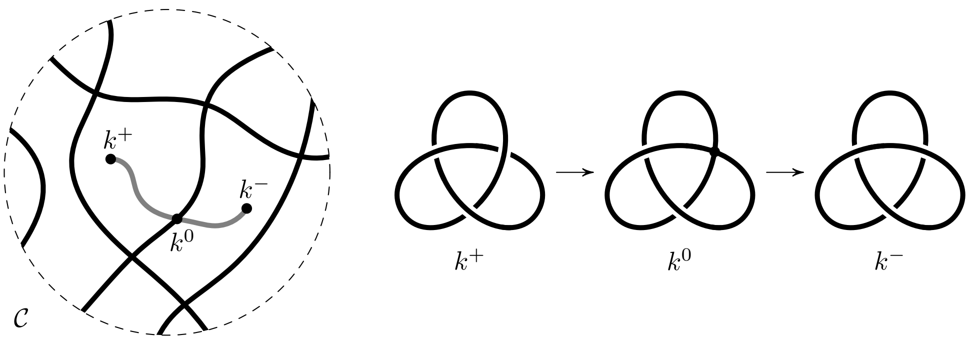
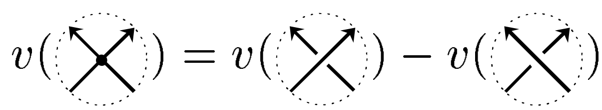



Figure 6.1. The crossing change that occurs when traversing $\Sigma_1$

A *Vassiliev invariant* is defined as a function $v : \Sigma \to \mathbb{C}$ that satisfies the *Vassiliev skein relation*

$$v(k^+) - v(k^-) = v(k^\circ)$$



where $k^o$ is an arbitrary singular knot with $n+1$ double points ($n \geq 0$), $k^+$ and $k_-$ are singular knots with $n$ double points obtained by desingularizing $k^o$ as shown in the figure. We say that $v$ is a Vassiliev invariant *of order* $\leq n$ if it satisfies the *finite-type condition*

$$v\Big|_{\Sigma_s} = 0 \quad \text{for any } s > n+1.$$

## 6.2. The one-term and four-term relations

The set $V_n$ of all Vassiliev invariants of order $\leq n$ has a linear space structure inherited from $\mathbb{C}$

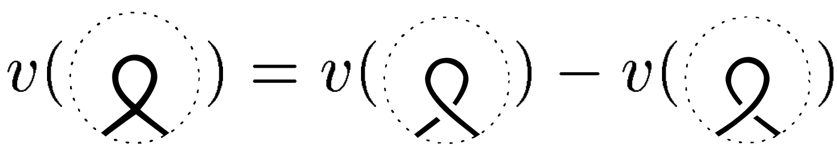
$$(v_1 + v_2)(k) = v_1(k) + v_2(k), \quad (\lambda v)(k) = \lambda \cdot v(k).$$

We have the following sequence of inclusions of linear spaces

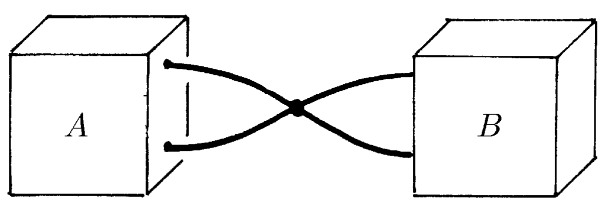
$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \subset V_\infty : \; = \bigoplus_{n=1}^{\infty} V_n/V_{n-1}.$$

Vassiliev invariants have the following properties:

(I) *One-term relation*:

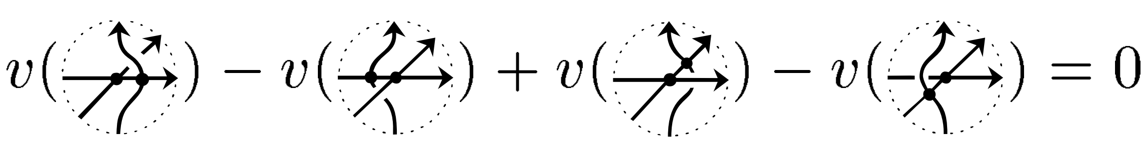


This easily follows from the Vassiliev skein relation if we assume (which we do!) that any $v \in V_n$, $n \geq 0$ is an ambient isotopy invariant, but it is more convenient for us to prove a slightly more general statement, namely that any Vassiliev invariant of any singular knot of the type pictured below is zero

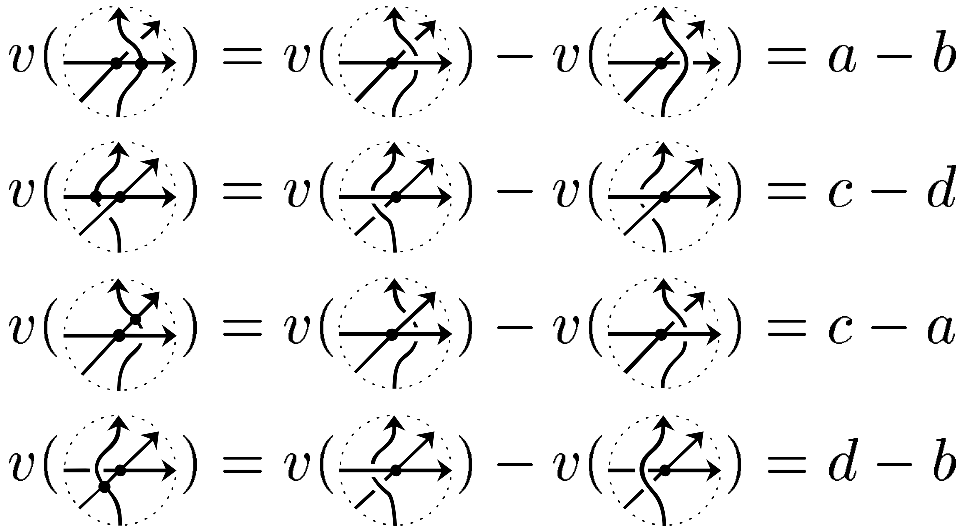



This is done by means of a lovely trick (see Exercise 5.1)

(II) *Four-term relation*:



To prove this, we apply the skein relation four times (using four different double points)




where the letters $a, b, c, d$ denote the value of the given Vassiliev invariant $v$ on identical ambient isotopic knots. Adding these four equalities, we obtain

$$(a - b - (c - d) + (c - a) - (d - b) = 0.$$

as claimed.

## 6.3. Dimensions of the spaces $V_n$

There is an infinite number of Vassiliev invariants. We shall learn how to compute the value of some of them (of small order) for concrete knots a little later, but first let us try to answer the following natural question: How "big" is the space $V_n$? Here we shall characterize its "size" by its dimension, which has been computed for $n \leq 12$. We present the dimensions of the linear spaces of Vassiliev invariants of order *strictly equal to* $n$, i.e., $\mathcal{V}_n := V_n/V_{n-1}$, for $n = 0, 1, 2, \ldots, 12$:

$$\dim \mathcal{V}_n = 1, 0, 1, 1, 3, 4, 9, 14, 27, 44, 80, 132, 232, \ldots$$

The proof is beyond the scope of this course, but we will find the dimensions of $\mathcal{V}_n$ for $n = 0, 1, 2, 3, 4$. To do this, we shall need the fundamental notion of chord diagram.

## 6.5. Chord diagrams

To each singular knot $k$, we asssociate a chord diagram as follows: the knot $k$ is an immersion of the circle $\mathbb{S}^1$ into $\mathbb{R}^3$; let us number the double points of $k$ in the order of their appearence as we go around the curve $k$, mark their two preimage points on $\mathbb{S}^1$ by the same number, and join each pair of identical numbers by chords; the obtained figure, denoted $D(k)$, is called the *chord diagram* of the singular knot $k$. An example of this construction is shown in Fig. 6.2.

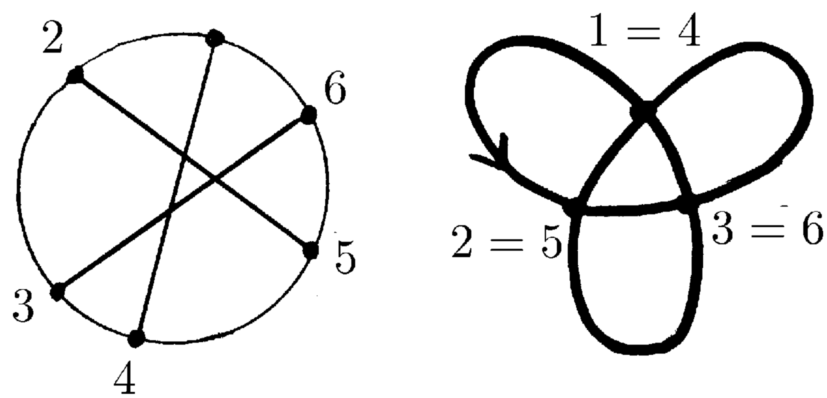



Figure 6.2. Example of a chord diagram

Two chord diarams are considered identical if there is a bijection between the chords preserving the order of their endpoints around the circle. Thus a chord diagram of a knot is well defined, it depends only on the knot and does not depend on the choice of double point from which we began our numbering. By definition, the chord diagram of any classical (i.e., nonsingular) knot is a circle without any chords.

We shall need the following lemma:

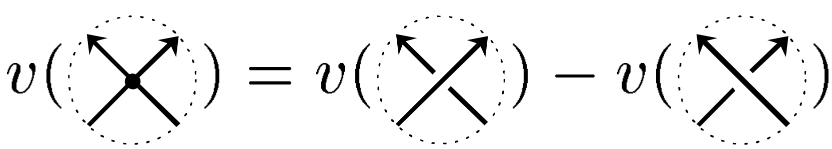
**Crossing Change Lemma 6.1.** *The value $v(k)$ of a Vassiliev invariant of order $n$ of a singular knot $k$ with $n$ crossings does not change under crossing changes.*

**Proof.** Since the order of $v$ is equal to the number of double points of $k$, the finite type condition and the skein relation together tell us that a crossing change "costs nothing".

It follows from Lemma 6.1 that a Vassiliev invariant of order $n$ of a singular knot $k$ with $n$ crossings depends only on the chord diagram of $k$, and so in that situation, we shall write $v(D(k))$ (where $D(k)$ is a picture of the chord diagram) instead of $v(k)$.

## 6.6. Vassiliev invariants of small order

We obviously have $\dim V_0 = 1$, because one can pass from any knot diagram to the unknot by a succession of crossing changes (Lemma 2.1), and these crossing changes "cost nothing", because in the particular case $n = 0$, the skein relation reads



where the singular knot on the left-hand side of the equation has exactly one double point and the two knots in the right-hand side have no singularities. Therefore, we can set $v(\bigcirc)$ equal to any real number, and it will satisfy the axioms of the theory. Thus $\dim V_0 = \dim \mathcal{V}_0 = 1$.

Now let $n = 1$. It turns out that we have

$$\dim(\mathcal{V}_1 = \dim(V_1/V_0) = 0,$$

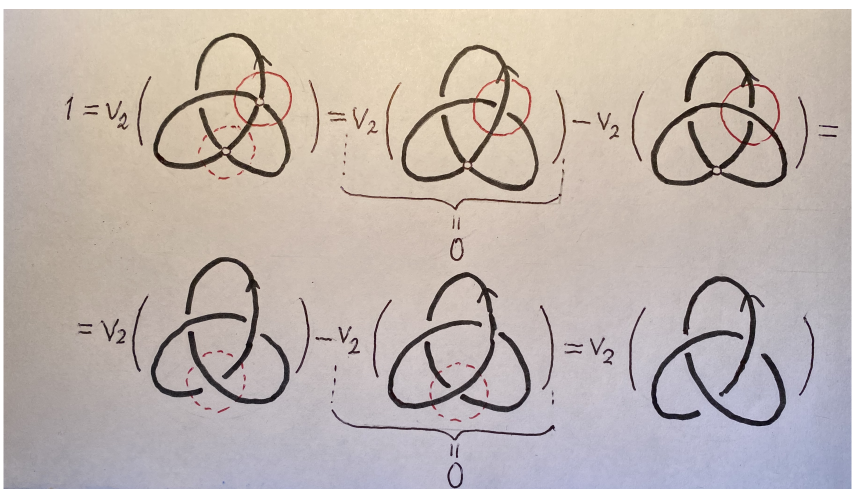
which means that $n = 1$ gives us no new invariants: all Vassiliev invariants of order 1 are actually of order 0. The proof of this disapointing result is the object of Exercise 6.4.

Fortunately, when we pass to $n = 2$, we obtain signficant results. Denote by $v_2$ the Vassiliev invariant of order 2 such that

$$v_2(\otimes) = 1, \quad v_2(\bigcirc) = 0$$

and also assume that $v_2$ vanishes on the chord diagram with two nonintersecting chords.

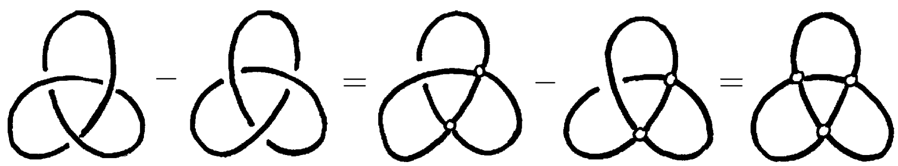
Let us compute the value of $v_2$ on the right trefoil $k$. By the Vassiliev skein relation, we have:



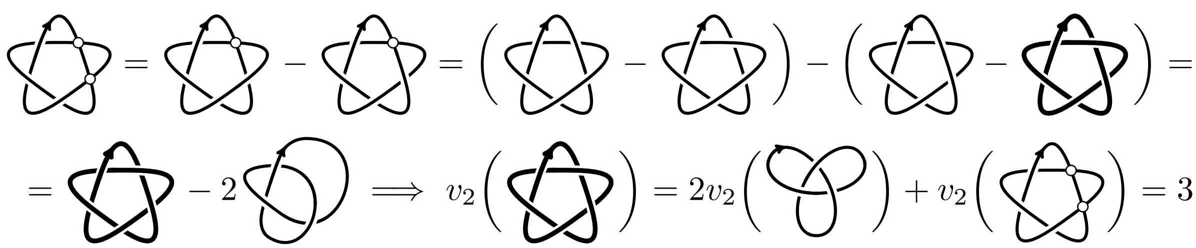

Thus we obtain $v_2(\text{righttrefoil}) = 1$. This is a meaningful result, since it implies that the trefoil is not the trivial knot.

In our further computations, we show pictures of the knots involved in the computation, but often omit writing $v_2(\quad)$ around each knot.

It is easy to show (Exercise 6.5) that the value of $v_2$ on the left trefoil is the same as that on the right one. However, Vassiliev invariants of order three already distinguish the right trefoil from the left one. It suffices to choose a Vassiliev invariant $v_3$ whose value on the chord diagram consisting of three pairwise intersecting chords is equal to 1. The computation that follows shows that the values of $v_3$ on the left and right trefoil differ by 1, so that the two trefoils are not isotopic:



Now let us compute the value of $v_2$ of the $5_1$ knot. We obtain



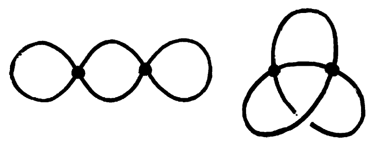
This shows that the knot $5_1$ is not isotopic to the trefoil. It also shows, together with the calculation in Exercise 6.5, that the knot $5_1$ is not isotopic to the eight knot $4_1$.

We are not going to learn how to compute the values of high order Vassiliev invariants. These computations, which involve the inductive construction of the so-called actuality tables, are rather cumbersome.

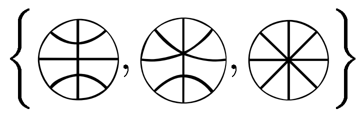
## 6.6. Exercises

**6.1.** Prove the one-term relation in its more general form.

**6.2.** Draw the chord diagrams of the following knots



**6.3.** Find singular knots with the following chord diagrams.



**6.4.** Prove that $\dim \mathcal{V}_1 = 0$

**6.5.** Calculate $v_2$ for the left trefoil.

**6.6.** Calculate $v_2$ for the eight knot.

**6.7.** Calculate $v_2$ for the $5_2$ knot.

**6.8.** Give a detailed proof of the fact that $\dim \mathcal{V}_2 = 1$.