Lecture 7

Combinatorial description of Vassiliev invariants

In the previous lecture, we defined Vassiliev invariants and learned how to compute their values for concrete knots in the simplest cases. To do this in more complicated situations, we need an additional rather intricate tool, called "actuality tables" (see CD-book, p.74). We will not study actuality tables, because our primary interest is not the practical computation of concrete values of $v(k) \in \mathbb{C}$, $v \in V_n$, $k \in \mathcal{K}$, but the study of the spaces V_n themselves. We shall see that they have a rich algebraic structure and possess a beautiful combinatorial description in terms of chord diagrams. We begin with some algebraic preliminaries.

7.1. Digression: graded algebras

By definition, an *algebra* A over $\mathbb{C}$ is a linear space (over $\mathbb{C}$) with a commutative associative operator (the *product*) $\cdot : A \times A \to A$ such that

$$(\alpha x+\beta y)\cdot(\gamma z) = \alpha\gamma(x\cdot z)+\beta\gamma(y\cdot z) \quad \forall \alpha, \beta, \gamma \in \mathbb{C}, \forall x, y, z \in A.$$

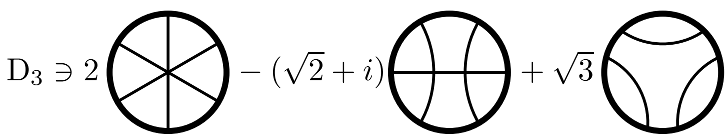
A *graded algebra* A over $\mathbb{C}$ is an algebra presented as the infinite sum of algebras $A = A_1 \oplus A_2 \oplus \dots \oplus A_n \oplus \dots$ satisfying the condition $A_p \times A_q \subset A_{p+q}$. The subscript n is the *grading*.

The simplest nontrivial example of a graded algebra is the algebra of polynomials $\mathbb{C}[x]$ with complex coefficients; for it, the product is the ordinary product of polynomials and the grading is the degree of the polynomial.

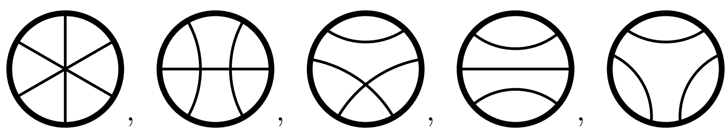
If the commutativity property of the algebra is replaced by skew commutativity, $xy = (-1)^{pq}\, yx$, $x \in A_p$, $y \in A_q$, then one obtains a *skewcommutative algebra*, important examples of which are differential forms and cohomology groups.

7.2. The graded algebra of chord diagrams

Linear combinations with coeffficients in $\mathbb{C}$ of chord diagrams with n chords have a natural linear space structure, denoted by D_n for any $n \geq 0$. Here is an example of an element of D_n:



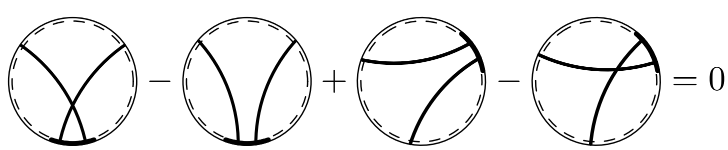
The dimension of D_3 is 5, because there are 5 different 3-chord diagrams



which form a basis for D_3.

Let D be the linear space $D = \bigoplus_{n=0}^{\infty} D_n$.

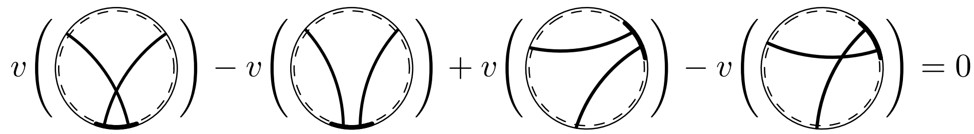
For any $n \geq 2$, we define the *four-term relation* for n-chord diagrams as



where only two of the n chords are shown in each of the n-chord diagrams, the remaining $n - 2$ chords (not shown) are exactly the same in all four diagrams and do not have any endpoints in the little fat arcs of the circles.

The use of the name "four term relation" is motivated by the following statement:

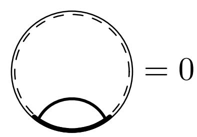
Lemma 7.1. *If $v \in V_n$, then*



where only two of the n chords are shown in the n-chord diagrams, the $n-2$ chords not shown are the same in all diagrams and do not have any endpoints in the little fat arcs of the circles.

The proof is the object of Exercise 7.2.

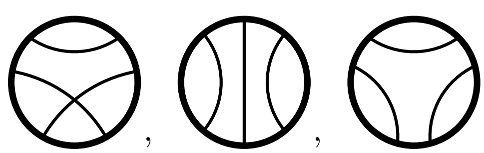
We now define the *one-term relation* for n-chord diagrams as



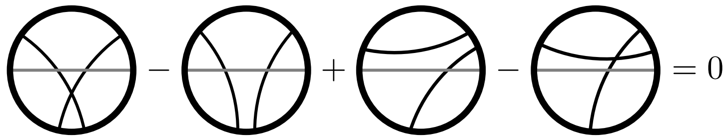
where only one chord is shown and the other $n-1$ chords have no endpoints in the fat arc of the circle. The use of the name "one-term relation" comes from the relation of the same name for Vassiliev invariants of knots.

We can now define the *algebra of chord diagrams* Δ as the linear space $\Delta = \bigoplus_{n=0}^{\infty} \Delta_n$, where each Δ_n is the quotient space of D_n by all possible one-term and four-term relations for n-chord diagrams; Δ is actually a *graded* algebra – the multiplication operation will be defined below, but first, as an illustration, we shall study Δ_3.

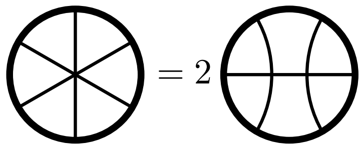
In Δ_3, the (equivalence classes of) the following three chord diagrams



are zero by the one-term relation. By the four-term relation, we have

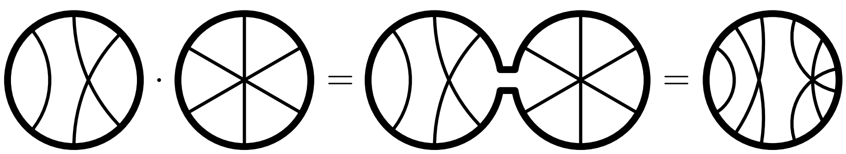


the third summand is zero by the one-term relation, and so



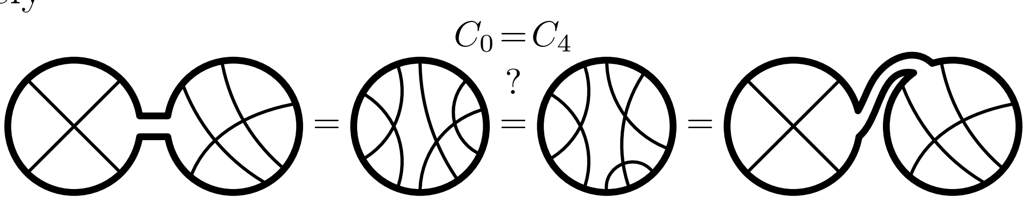
which means that $\dim \Delta_3 = 1$.

We now define the product of two (equivalence classes of) chord diagrams $C_1 \in \Delta_p$, $C_2 \in \Delta_q$ as follows:

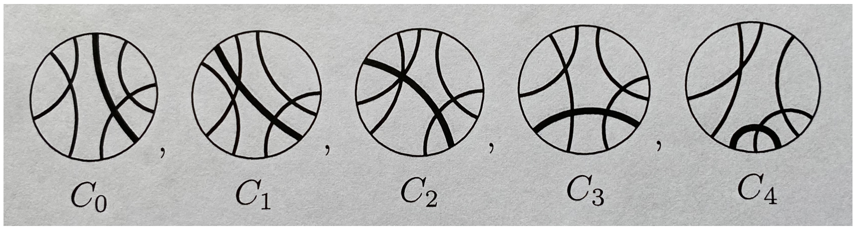


Lemma 7.2. *The above product is well defined, i.e., it does not depend on the choice of representatives and of the places on the circles where the circles are glued together.*

Sketch of the proof. The independance of the choice of representatives is obvious. Let us prove that the product does not depend on the choice of the place where the circles are glued together. We will do this in the case of a concrete example, namely



To prove that the 5-chord diagrams $C_0, C_4 \in D_5$ are equivalent, we move the upper endpoint of the fat chord in C_0 counterclockwise successively over the next endpoint four times, obtaining the chord diagrams C_1, C_2, C_3, C_4:



Applying the four-term relation to C_0, C_1, C_2, C_3 and then to C_1, C_2, C_3, C_4, we see that

$$C_0 - C_1 + C_2 - C_3 = 0 \quad \text{and} \quad C_1 - C_2 + C_3 - C_4.$$

Adding the last two equalities, we obtain $C_0 = C_4$, as required.

In the gereral case, the argument is similar.

Remark 7.1. The algebra of chord diagrams is actually a *bialgebra*: besides the multiplication defined above, it possesses a *comultiplication*. We will not define this additional structure (see the CD-book, p.92).

7.3. The Vassiliev–Kontsevich theorem

Let Δ_n be the algebra of n-chord diagrams regarded as a linear space; we denote by Δ_n^* its *dual space* [1], i.e., the space of linear functions on elements of Δ_n; recall that V_n denotes the linear space of Vassiliev invariants of order $\leq n$.

Theorem 7.1. *For any $n \geq 0$, there exists an isomorphism*

$$\alpha_n \colon\ V_n/V_{n-1} \to \Delta_n^*.$$

We will not prove this deep theorem, which gives a simple combinatorial description of the space of Vassiliev invariants, but will construct the map α_n. Let v_n be a Vassiliev invariant of order exactly equal to n. Its image must be a linear function

[1] The space Δ_n^* is denoted by CD_n^* in the CD-book, where it is called the "space of unframed weight systems".

$l : \Delta_n \to \mathbb{C}$; by linearity it suffices to define l on a basis $d_1, \ldots, d_s$ of Δ_n. We set

$$l(d_i) = v_n(d_i) \quad \text{for all} \quad i \in \{1, \ldots, s\},$$

(The fact that the right-hand side of this equality is well defined easily follows from the Crossing Change Lemma.)

The injectivity of α_n is the object of Exercise 7.5, its surjectivity can be proved by using the Kontsevich integral, which we will study in the next lecture.

Recall that $\Delta = \bigoplus_{n\geq 0} \Delta_n$ is a graded algebra and therefore $\Delta^* = \bigoplus_{n\geq 0} \Delta_n^*$ inherits this structure via α_n^{-1}.

Thus for all $n \geq 0$, the space $\mathcal{V} = \bigoplus_{n\geq 0} V_n/V_{n-1}$ of Vassiliev invariants is a graded algebra.

7.4. Vassiliev invariants vs. other invariants

It turns out that the space of all Vassiliev invariants is more powerful that any of the other previously known knot invariants: in fact, the previously known invariants can all be expressed in terms of Vassiliev invariants. In this section, we will state three such results without proof.

Fact 7.1. *The n-th coefficient of the Conway polynomial is a Vassiliev invariant of order $\leq n$.*

In the Jones polynomial of a knot, let us substitute $q = e^h$, expand it into an infinite power series in h and denote by j_n the coefficient at h^n.

Fact 7.2. *The coefficient j_n in the power series defined above is a Vassiliev invariant of order $\leq n$.*

The *Casson invariant* C is a classical integer-valued invariant of homology 3-spheres. (Each homology sphere is obtained from the ordinary 3-sphere by surgery along a knot K, so that C can be regarded as a knot invariant.)

Fact 7.3. *The Casson invariant C is equal to the second coefficient of the Conway polynomial and is therefore a Vassiliev invariant of order ≤ 2.*

For the proofs, see the CD-book.

7.5. Exercises

7.1. Find the dimensions of the linear spaces D_2 and D_4.

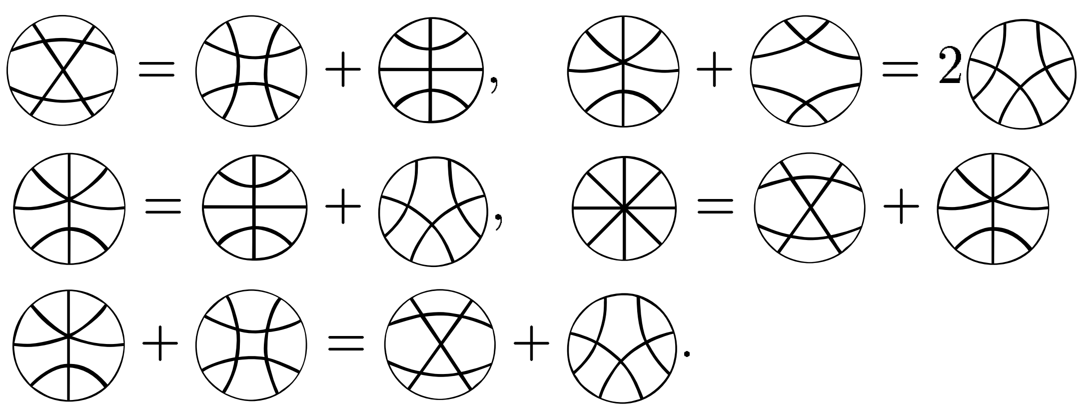
7.2. Prove Lemma 7.1.

7.3.. Find the dimension of Δ_4.

7.4. Find the dimension of Δ_5.

7.5. Prove that the map α_n is injective.

7.6. Prove the first two of the following relations



7.7. Prove the last three of the relations in the picture above.