Reflection Equation algebras and their applications

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Plan

Consider the Lie algebra $gl(N)$ and its enveloping algebra $U(gl(N))$. Also, consider the super-algebra $gl(m|n)$ and its enveloping algebra $U(gl(m|n))$. What is the main difference between these algebras? The algebra $gl(N)$ is defined in the space $End(V)$ as follows

$$[X, Y] = X \circ Y - \circ P(X \otimes Y) \in End(V),$$

where $P(X \otimes Y) = Y \otimes X$ is the usual flip.
Whereas for the bracket in the super-algebra $gl(m|n)$ the flip is replaced by a super-flip $P_{m|n} : V^{\otimes 2} \to V^{\otimes 2}$, where $V = V_0 \oplus V_1$ is $\mathbb{Z}_2$-graded space and

$$P_{m|n}(a \otimes b) = (-1)^{\overline{a}\overline{b}} b \otimes a.$$

Amusing fact: the relations in the algebras $gl(N)$ and $gl(m|n)$ can be written in the following matrix form. Let $\{m_i^j\}$ be the standard basis in the algebra $End(V)$, that is $m_i^j \triangleright x_k = x_i\, \delta_k^j$, where $\{x_k\}$ is a basis in the basic space V.
Compose the matrix $M = (m_i^j)_{1\le i,j\le N}$ and consider the following systems

$$P\, M_1\, M_2 - M_1\, M_2\, P = 0,\;\; M_1 = M \otimes I,\;\; M_2 = I \otimes M$$

and

$$P\, M_1\, M_2 - M_1\, M_2\, P = M_2 - M_1.$$

The former system written via the entries reads

$$m_i^j\, m_k^l = m_k^l\, m_i^j,\;\; \forall\, i, j, k, l,$$

i.e. the entries commute with each other.

The latter system reads

$$m_i^j\, m_k^l - m_k^l\, m_i^j = m_i^l\, \delta_k^j - m_k^j\, \delta_i^l.$$

This is the system of relations in the enveloping algebra $U(gl(N))$.
Example: $V = span(x, y)$, $V^{\otimes 2} = span(x \otimes x, x \otimes y, y \otimes x, y \otimes y)$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$M_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad M_2 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

The corresponding homogenous system reads

$$ab = ba,\ ac = ca, \ldots$$

The corresponding inhomogeneous system reads

$$ab - ba = b,\ ac - ca = -c, \ldots$$

Now, we want to get similar systems for super-algebras. It is possible to do in different ways. In order to get super-commutative algebra, we constitute the following system

$$P_{m|n}\, M_1\, P_{m|n}\, M_1 - M_1\, P_{m|n}\, M_1\, P_{m|n} = 0.$$

It means that in the matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the sub-matrices A and D are composed from even elements and these B and C- from odd ones.

As for the relations in the $U(gl(m|n))$ they can be presented as

$$P_{m|n}\, M_1\, P_{m|n}\, M_1 - M_1\, P_{m|n}\, M_1\, P_{m|n} = P_{m|n}\, M_1 - M_1\, P_{m|n}.$$

By writing this system via entries we get the defining relations in $U(gl(m|n))$.
Another way of constructing an algebra with super-commuting generators is

$$P_{m|n}\, M_1\, M_2 - M_1\, M_2\, P_{m|n} = 0.$$

Since $M_2 = P\, M_1\, P$, it is possible to write down it as follows

$$P_{m|n}\, M_1\, P\, M_1\, P - M_1\, P\, M_1\, P\, P_{m|n} = 0.$$

Here, the operators P and $P_{m|n}$ are mixed and the blocs A, B, C, D cannot be given the same meaning.
Below, the super-flip $P_{m|n}$ will be replaced by other operators.

Let us introduce some symmetric polynomials of M (namely, elementary ones and power sums)

$$\det(M - t\,I) = \sum_0^N (-t)^{N-k} e_k(M), \quad p_k(M) = Tr\, M^k.$$

If M is a triangular matrix the elements $e_k(M)$ and $p_k(M)$ are respectively elementary symmetric polynomials and power sums in the eigenvalues μ_i of M. Namely, we have

$$e_k = \sum_{i_1<...<i_k} \mu_{i_1}...\mu_{i_k}, \quad p_k(M) = \sum \mu_i^k.$$

Also, note that these symmetric polynomials of M are related by the Newton identities

$$k\, e_k - p_1\, e_{k-1} + p_2\, e_{k-2} + \cdots + (-1)^k p_k\, e_0 = 0.$$

Now, deform $P \to R$ in the corresponding systems–homogeneous and not. And do the same with the super-flip $P_{m|n}$. Namely, take R as follows (here $N = 2, \quad m = n = 1$)

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

Note that for $q \to 1$ we respectively recover the flip P and the super-flip $P_{1|1}$.

If we deform the system $P\, M_1\, P\, M_1 - M_1\, P\, M_1\, P = 0$ and its inhomogeneous analog, we get

$$R\, M_1\, R\, M_1 - M_1\, R\, M_1\, R = 0.$$

$$R\, M_1\, R\, M_1 - M_1\, R\, M_1\, R = R\, M_1 - M_1\, R.$$

The first one will be called Reflection Equation (RE) algebra. The second one–modified RE algebra.

If we deform P in the system $P\, M_1\, M_2 - M_1\, M_2\, P = 0$, we get

$$R\, M_1\, M_2 - M_1\, M_2\, R = 0 \;\Leftrightarrow\; R\, M_1\, P\, M_1\, P - M_1\, P\, M_1\, P\, R = 0.$$

This algebra will be called RTT algebra.

Note that all these algebras make sense for some other operators R. Question: what operators $R : V^{\otimes 2} \to V^{\otimes 2}$ are "good"?

We call an invertible linear operator $R: V^{\otimes 2} \to V^{\otimes 2}$ *braiding* if it satisfies the so-called *braid relation*

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I, \;\; R_{23} = I \otimes R.$$

Then the operator $\mathcal{R} = R\,P$ where P is the usual flip is subject to the QYBE

$$\mathcal{R}_{12}\, \mathcal{R}_{13}\, \mathcal{R}_{23} = \mathcal{R}_{23}\, \mathcal{R}_{13}\, \mathcal{R}_{12}.$$

A braiding R is called *involutive symmetry* if $R^2 = I$.
A braiding is called *Hecke symmetry* if it is subject to the Hecke condition

$$(q\,I - R)(q^{-1}\,I + R) = 0, \;\; q \in \mathbb{C}, \;\; q \neq 0, \;\; q \neq \pm 1.$$

In particular, such a symmetry comes from the QG $U_q(sl(N))$. For $N = 2$ it is just the example above.

We assume q to be generic. This means that

$$k_q = \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0 \text{ for any integer } k.$$

As for the braidings coming from the QG of other series B_n, C_n, D_n, each of them has 3 eigenvalues and it is called BMW symmetry.

In order to classify Hecke symmetries, consider "R-symmetric" and "R-skew-symmetric" algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \quad \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

where $T(V)$ is the free tensor algebra. Also, consider the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \quad P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V)t^k,$$

where the upper index (k) labels homogenous components of these quadratic algebras.
If R is involutive, we put $q = 1$ in these formulae.

The following holds $P_-(-t)P_+(t) = 1$.

Proposition. (Phung Ho Hai)

The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_r\, t^r}{1 - b_1\, t + \ldots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all involutive and Hecke symmetries are similar to super-flips, for which the role of the bi-rank is played by the super-dimension $(m|n)$.

Examples. If R comes from the QG $U_q(sl(m))$, then

$$P_-(t) = (1+t)^m.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $N \geq 2$ there exits a Hecke symmetry such that

$$P_-(t) = 1 + Nt + t^2.$$

Here $dim V = N$, the bi-rank is $(2|0)$.
If $P_-(t)$ is a polynomial, i.e. the bi-rank of R is $(m|0)$, R is called *even*.

Thus, for a given an involutive or Hecke symmetry R, we consider 3 quantum matrix algebras. If $T = (t_i^j)$ is subject to the system

$$RT_1T_2 - T_1T_2R, \;\; T = (t_i^j), 1 \le i, j \le m,$$

is called RTT algebra. If L is subject to

$$RL_1RL_1 - L_1RL_1R = 0, \;\; L = (l_i^j), \;\; 1 \le i, j \le m,$$

the algebra is called RE one. And if the system is

$$RL_1RL_1 - L_1RL_1R = RL_1 - L_1R, \;\; L = (l_i^j), \;\; 1 \le i, j \le m$$

the algebra is called modified RE algebra.

Introduce the following notations

$$L_{\overline{1}} = L_1, \;\; L_{\overline{2}} = R_{12}\, L_{\overline{1}}\, R_{12}^{-1}, \;\; L_{\overline{3}} = R_{23}\, L_{\overline{2}}\, R_{23}^{-1} = R_{23}\, R_{12}\, L_{\overline{1}}\, R_{12}^{-1}\, R_{23}^{-1}, \ldots$$

In this notation the defining relations of the RE algebra become similar to the RTT ones

$$RL_{\overline{1}}L_{\overline{2}} = L_{\overline{1}}L_{\overline{2}}R.$$

Similarly to the classical case, in the RE algebra it is possible to define analogs of the power sums

$$p_k(L) = Tr_R\, L^k.$$

Here, Tr_R is the so-called quantum or R-trace, which is defined by the formula

$$Tr_R M = Tr\, C_R\, M.$$

The matrix C_R is a matrix which is completely defined by the initially given involutive or Hecke symmetry R and can be constructed for almost any such a symmetry.
Also, analogs $e_k(L)$ of elementary symmetric polynomials can be introduced.

Note that if R is of bi-rank $(m|0)$, the highest elementary symmetric polynomial $e_m(L)$ is the "quantum determinant".

Observe that the elementary symmetric polynomials $e_m(L)$ and power sums $p_k(L)$ are central in the RE algebra. Moreover, as shown in [GPS, IOP], in the RE algebra there is a version of the Newton identities

$$p_k - q p_{k-1}\, e_1 + (-q)^2 p_{k-2}\, e_2 + \ldots + (-q)^{k-1} p_1\, e_k + (-1)^k k_q e_k = 0$$

Also, in this algebra there exists a quantum analog of the Cayley-Hamilton identity similar to the classical one

$$L^m - q\, L^{m-1}\, e_1 + (-q)^2 L^{m-2}\, e_2 + \ldots + (-q)^{m-1} L\, e_{m-1} + (-q)^m\, I\, e_m = 0.$$

Now, we consider algebras useful for integrable systems. First, introduce braidings depending on parameters.

Claim 1. If R is an involutive symmetry, then

$$R(u,v) = R - \frac{a\,I}{u-v}$$

meets the quantum Yang-Baxter equation

$$R_{12}(u,v)\,R_{23}(u,w)\,R_{12}(v,w) = R_{23}(v,w)\,R_{12}(u,w)\,R_{23}(u,v).$$

Claim 2. If $R = R(q)$ is a Hecke symmetry, then the same is valid for

$$R(u,v) = R(q) - \frac{(q-q^{-1})u\,I}{u-v}.$$

The Drinfeld's Yangian $\mathbf{Y}(gl(N))$ is in fact an RTT algebra defined by

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(v)\,T_2(u)\,R(u,v)$$

with the Yang braiding $R(u,v) = P - \frac{a\,I}{u-v}$. The matrix $T(u)$ is usually developed in a series

$$T(u) = \sum_{k\geq 0} T[k]\,u^{-k}.$$

Besides, one assumes $T[0] = I$.

Introduce two types of GY in a similar manner.

1. Generalized Yangians of RTT type are defined by

$$R(u,v)T_1(u)\,T_2(v) = T_1(v)\,T_2(u)\,R(u,v),$$

where $R(u,v)$ is one of the above current braidings.
2. GY of RE type (also called braided Yangians) are defined by

$$R(u,v)L_{\overline{1}}(u)\,L_{\overline{2}}(v) = L_{\overline{1}}(v)\,L_{\overline{2}}(u)\,R(u,v).$$

Here $L_{\overline{2}}(v) = R\,L_{\overline{1}}(v)\,R^{-1}$.
These relations can be also presented as follows

$$R(u,v)L_1(u)RL_1(v) = L_1(v)RL_1(u)R(u,v).$$

If a braiding $R(u, v)$ arises from an involutive symmetry R, the corresponding GY is called *rational*. If R is Hecke, then it is called *trigonometrical*.

If R is of bi-rank $(m|0)$, we define quantum determinants in GY of RE type in the rational (resp., trigonometrical) case as follows

$$\det(L(u)) =< v|L_{\overline{1}}(u)\, L_{\overline{2}}(u-1)\dots L_{\overline{m}}(u-m+1)|u>,$$

$$\det(L(u)) =< v|L_{\overline{1}}(u)\, L_{\overline{2}}(q^{-2}u)\dots L_{\overline{m}}(q^{-2(m-1)}u)|u>,$$

where u and v are some vectors.
In the GY of RTT type the formulae are similar but the indexes are not overlined.
Observe that arguments of the $L(u)$ enter these formulae with shifts, additive in the rational cases and multiplicative in the trigonometrical ones.

The quantum elementary symmetric polynomials $e_k(u)$ in GY of RE type are in the trigonometrical case

$$e_k(u) = Tr_{R(1\dots k)}\left(A_R^{(k)} L_{\overline{1}}(u) L_{\overline{2}}(q^{-2}u)\dots L_{\overline{k}}(q^{-2(k-1)}u)\right),\ k \geq 1,$$

The quantum power sums are defined in this case by

$$p_k(u) = Tr_R\, L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)...L(u).$$

Let us exhibit the quantum Newton relations in the GY of RE type in the trigonometrical case

Proposition.

$$p_k(u) - q p_{k-1}(q^{-2}u)e_1(u) + (-q)^2 p_{k-2}(q^{-4}u)e_2(u) + \dots$$
$$+(-q)^{k-1} p_1(q^{-2(k-1)}u)e_k(u) + (-1)^k k_q e_k(u).$$

Also, there is an analog of the Cayley-Hamilton identity.
Note that the determinant in the GY of RE type is central and all elementary symmetric polynomials and power sums form a Bethe subalgebra. Namely, this property is useful for constructing integrable systems.

Let A and B be two associative unital algebras equipped with a map (called permutation map)

$$\sigma : A \otimes B \to B \otimes A, \ \ (a \otimes b) \to \sigma(a \otimes b), \ \ a \in A, \ \ b \in B \qquad (1)$$

Definition.

By a quantum double (QD) we mean the data (A, B, σ), where the map σ is defined by means of a braiding R.

Now, introduce the following QD

$$R\, M_1\, R\, M_1 = M_1\, R\, M_1\, R,$$

$$R^{-1}\, D_1\, R^{-1}\, D_1 = D_1\, R^{-1}\, D_1\, R^{-1},$$

$$D_1\, R\, M_1\, R = R\, M_1\, R^{-1}\, D_1 + R.$$

The last relation arises from the permutation map and we call it permutation relation.

If $R \to P$, this QD tends to the usual Weil-Heisenberg algebra.

In general we have a quantum Weil-Heisenberg algebra. Note that the entries of the matrix $D = \|\partial_i^j\|$ are treated to be quantum analogs of the patrial derivatives. In order to get an action of the elements ∂_k^l onto these m_i^j, we permute these elements

$$\partial_k^l \otimes m_i^j \to \sigma(\partial_k^l \otimes m_i^j)$$

and kill all partial derivatives located on the right hand side. For instance, in the classical case we have

$$\partial_k^l \triangleright m_i^j = \delta_k^j \, \delta_i^l.$$

Theorem.

Let $L = \|l_i^j\|_{1\le i,j\le N}$ and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be matrices entering the above QD. Then the matrix

$$\hat{L} = L\, D$$

generates the modified RE algebra.

This theorem says that in our quantum setting, the situation is similar to the classical one. Recall that in the classical setting the matrix $\hat{L} = L\, D$, where L is a matrix with commutative entries l_i^j and D is the matrix composed from the partial derivatives $\partial_i^j = \partial_{l_j^i}$, generates the algebra $U(gl(N))$. This is a sort of bosonization.

However, in the quantum case there is a property, which is absent in the classical situation. If the matrix $\hat{L}$ is subject to the modified RE algebra, then the matrix

$$L = I - (q - q^{-1})\hat{L}$$

meets the non-modified RE algebra. Thus, the RE algebra and its modified version are in fact one algebra but written in different bases.

Note that this isomorphism fails if $q \to 1$. Since the RE algebra tends to $Sym(gl(N))$ and the modified RE algebra tends to $U(gl(N))$, we get two algebras for which we have no isomorphism.

However, if $q \neq \pm 1$ the above isomorph is valid and in the above QD we can pass to a modified RE algebra instead of a non modified one. In this way we can define analogs of partial derivatives on the modified RE algebra. By passing to the limit $q = 1$ we get analogs of partial derivatives on the algebra $U(gl(N))$.
Consider an example $N = 2$. By passing to the compact form of the algebra $U(gl(2))$. i.e. to $U(u(2))$, we get the generators x, y, z, t with commutations $[x, y] = z$... etc.
Permutations of these generators and the partial derivatives are $\partial_x\, y - y\, \partial_x = \frac{1}{2}\, \partial_z$ etc. So, in order to apply ∂_x to for instance yz, we have to permute ∂_x with y and z and kill all derivatives located on the most right position. We have $\partial_x(yz) \neq 0$.

Now, go the classical Capelli identity. Let $\hat{L} = L\,D$. Then we have

$$rDet(\hat{L} + K) = detL\, detD,$$

where K is the diagonal matrix $diag(0, 1, ..., n-1)$ and $rDet$ is the so-called row-determinant.
Observe that the term $rDet(\hat{L} + K)$ in the l.h.s. can be written in the following form

$$cDet(\hat{L} + K) = detL\, detD,$$

where K is the diagonal matrix $diag(n-1, ...1, 0)$ and $cDet$ is the so-called column-determinant. Also, the following form

$$Tr_{1..N}A^{(N)}\hat{L}_1\,(\hat{L} + I)_2\,(\hat{L} + 2I)_3...(\hat{L} + (N-1)I)_N$$

is possible.

Proposition.

In the RE algebra the following holds

$$Tr_{R(1\ldots m)}\,A^{(m)}\,\hat{L}_1\,(\hat{L}_{\overline{2}}+q\,I)(\hat{L}_{\overline{3}}+q^2\,2_q\,I)\ldots(\hat{L}_{\overline{m}}+q^{m-1}\,(m-1)_q\,I)=$$

$$q^{-m}\,det_R\,L\,det_{R^{-1}}\,D.$$

Here m is the rank of R. (Note that in the classical case $m=N$.)

Whereas the determinants in the r.h.s. are defined by the formulas

$$det_R\,L=\langle v|\,L_1\,L_{\overline{2}}\ldots L_{\overline{m}}\,|u\rangle,$$

$$det_{R^{-1}}\,D=\langle v|\;D_{\overline{m}}\,D_{\overline{m-1}}\ldots D_1\,|u\rangle.$$

Observe that there are known numerous attempts to generalize the classical Capelli identity.
I want to only mention the paper by Noumi, Umeda, Wakayama (1994). Their construction is related to the RTT algebra. Their R is the standard Hecke symmetry, i.e., it comes from the QG $U_q(sl(N))$. Whereas ours is valid in general situation.

In 1996, A.Okounkov introduced the notion of quantum immanants. Our technique enables us to introduce q-analogs of these objects.

Now, recall the representation theory of the algebra $U(gl(N))$. Let

$$\pi_\lambda : U(gl(N)) \to \mathrm{End}(V_\lambda)$$

be an irreducible finite dimensional representations of the algebra $U(gl(N))$. Here

$$\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0)$$

is a partition of an integer k. It is known that equivalence classes (up to conjugations) of representations are labeled by such partitions.

More precisely, there exists an idempotent P_λ belonging to the group algebra $\mathbb{C}[S_k]$ of the symmetry group such that the subspace $P_\lambda\, V^{\otimes k}$ is that of such an irreducible representation. By the Schur lemma, the image of any central element of the algebra $U(gl(N))$ in this subspace is a scalar operator.
Examples of central elements are those arising from the power sums of the generating matrix. They are called Casimir operators.
Their eigenvalues have been computed by Perelomov and Popov.

Recall that we consider the modified RE algebra $\hat{\mathcal{L}}(R)$ to be a q-analog of the enveloping algebra $U(gl(N))$. Also, a representation theory of the algebra $\hat{\mathcal{L}}(R)$ is similar to that of $U(gl(N))$. More precisely, there exist analogs $P_\lambda(\mathrm{q})$ of the idempotents P_λ belonging to the Hecke algebra such that $P_\lambda(q)\, V^{\otimes k}$ are irreducible representations of the algebra $\hat{\mathcal{L}}(R)$.

Analogs of the above Casimir operators can also be introduced by $Tr_R\hat{L}^k$. We have computed their eigenvalues for small k and l and formulated a conjecture about those in general.

Also, by using this technique we introduced q-analogs of the so-called cut-and-joint operators which are intensively used in Hurwitz theory.

According to a Frobenius theorem any numerical matrix M can be reduced by transformations

$$M \mapsto g\, M\, g^{-1}$$

to the so-called second canonical form

$$M_{can} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_m & a_{m-2} & a_{m-1} & \dots & a_1 \end{pmatrix}.$$

It is clear that

$$a_1 = Tr\, M, \dots, a_m = (-1)^{N-1}\, det\, M.$$

Consider a connection operator

$$\partial_u - M(u),$$

where $M(u)$ is a numerical matrix smoothly depending on a parameter u.
According to the famous DS theorem this operator can be reduced to that

$$\partial_u - M_{can}(u),$$

where $M_{can}(u)$ has the forme above, by gauge transformations

$$\partial_u - M(u) \mapsto g(u)\,(\partial_u - M(u))\,g(u)^{-1}.$$

Also, Drinfeld-Sokolov identified the reduced Poisson structure corresponding to $\widehat{gl(N)}$ with the second Gelfand-Dikey structure.

The question: what is q-analog of this reduction?

Frenkel-Reshetikhin-STS suggested to replace ∂_u by the operator $D_q\, f(u) = f(qu)$ and the above gauge transformations by

$$D_q - M(u) \to g(qu)\,(D_q - M(u))\,g(u)^{-1}.$$

Then a similar claim is valid.
Observe that F-R-STS are dealing with numerical matrix $M(u)$.

We are dealing with matrices whose entries are not commutative. For instance consider the generating matrix $M = (m_i^j)$ of the algebra $U(gl(N))$. Then there exists a polynomial

$$p(t) = t^N + a_{N-1}\, t^{N-1} + \dots + a_0\, 1 = 0$$

where the coefficients are central in the algebra $U(gl(N))$, such that $p(M) = 0$. This is NC version of the CH identity.

Proposition.

There exists a matrix

$$M_{can} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_N & a_{N-2} & a_{N-1} & \dots & a_1 \end{pmatrix}.$$

where the entries a_k are elements of $U(gl(N))$, such that M and M_{can} are similar in the following sense: there exists a nontrivial matrix C with entries from $U(gl(N))$ such that

$$C\,M = M_{can}\,C.$$

A similar claim is valid in all QM algebras where there exists an analog of the CH relation for the generating matrix (i.e. $U(gl(m|n))$, RE algebras (modified or not), Generalized Yangians of RE type). We exhibit this claim for in the trigonometric Generalized Yangians of RE type.

Consider the operator $L(u)q^{2\,u\,\partial_u}$, $\partial_u = \frac{d}{du}$ and the matrix

$$L_{can}(u) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_N(u) \\ 1 & 0 & \dots & 0 & 0 & a_{N-1}(u) \\ 0 & 1 & \dots & 0 & 0 & a_{N-2}(u) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & a_2(u) \\ 0 & 0 & \dots & 0 & 1 & a_1(u) \end{pmatrix},$$

$$a_k(u) = -(-q)^k\, e_k(q^{2(N-1)}u)$$

Proposition.

There exists a nontrivial matrix $C(u)$ such that the following holds

$$(L(u)q^{2\,u\,\partial_u})C(u) = C(u)(L_{can}(u)q^{2\,u\,\partial_u}).$$

This proposition and all similar ones can be shown by the Chervov-Talalaev method.

By concluding my talk I want to compare the objects introduced by Drinfeld and ours. In the middle 80's Drinfeld introduced the notion of QG. The simplest one is $U_q(sl(2))$, generated by 3 generators X, Y, H subject to

$$HX - XH = 2X, \; HX - XH = -2Y, \; XY - YX = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

However in fact, this algebra is isomorph to the (completed) enveloping algebra $U(sl(2))$. Only the coalgebraic structure is deformed. Whereas our modified RE algebra is a deformation of the algebra $U(gl(N))$ indeed (provided R is a deformation of P). Moreover, it is possible to treat our modified RE algebra as a two-parameter deformation of $Sym(gl(N)))$.

This two-parameter deformation is a quantization of a Poisson pencil

$$a\{\, ,\, \}_{gl(N)} + b\{\, ,\, \}_{RE},\ \ a, b \in \mathbb{C},$$

where the bracket $\{\, ,\, \}_{gl(N)}$ is the linear Poisson-Lie bracket. For $N = 2$ its restriction to $sl(2)$ is

$$\{h, x\} = 2\, x,\ \{h, y\} = -2\, y,\ \{x, y\} = h.$$

The bracket $\{\, ,\, \}_{RE}$ is quadratic. If $N = 2$, its restriction to $sl(2)$ is

$$\{h, x\} = x\, h, \{h, y\} = -y\, h,\ \{x, y\} = h^2.$$

Many thanks