Polynomial algebra, affine space, and projective space associated with a vector space.

POLYNOMIALS. Let V be n-dimensional vector space over an arbitrary field k, $(e_1, e_2, \dots, e_n)$ be a basis in V, and $(x_1, x_2, \dots, x_n)$ be the corresponding dual basis[1] in V^*. There is *the evaluation map*

$$k[x_1, \dots, x_n] \xrightarrow{ev} \{k\text{-valued functions on } V\}$$

which sends a polynomial $f(x_1, x_2, \dots, x_n)$ to the function

$$p = (p_1, p_2, \dots, p_n) \longmapsto f(p) = f(p_1, p_2, \dots, p_n)\,.$$

The image of ev is called *a space of algebraic functions* on V.

Lemma 1.1 *The evaluation map is injective*[2] *iff k is infinite.*

Proof. If k is finite, say, consists of q elements, then the space of k-valued functions on V consists of q^{q^n} elements and is finite as well. Since the polynomial algebra is infinite, $\ker ev \neq 0$. The inverse argumentation uses induction on n. Non zero polynomial $f(x)$ in one variable can't have more than $\deg f$ roots and has to be zero as soon as it vanishes at infinitely many points of k. Polynomial in n variables can be considered as a polynomial in one variable x_n with the coefficients from the algebra $k[x_1, \dots, x_{n-1}]$:

$$f(x_1, x_2, \dots, x_n) = \sum_{\nu=0}^{d} \varphi_\nu(x_1, x_2, \dots, x_{n-1}) \cdot x_n^{d-\nu}\,.$$

Evaluating coefficients at any point $(p_1, p_2, \dots, p_{n-1}) \in k^{n-1}$, we get the polynomil in one variable x_n with the constant coefficients. It equals zero by the above reason. Hence all the coefficients φ_ν are identically zero functions and have to be zero polynomials by the inductive assumption. $\square$

SYMMETRIC ALGEBRA. $k[x_1, \dots, x_n]$ is *the graded algebra*, i. e. as a vector space it is decomposed into the direct sum of subspaces $k[x_1, \dots, x_n] = \bigoplus\limits_{d \geqslant 0} V_d$ such that $V_r V_s \subset V_{r+s}$, where V_d consists of all homogeneous polynomials of degree d. The subspaces V_d don't depend on the choice of coordinates and can be canonically recovered from V. For example, $V_0 = k$ is just the ground field and $V_1 = V^*$ is naturally identified with the dual space of V. Other V_d are canonically isomorphic to *symmetric powers* $S^d(V^*)$ that are defined by some "universal mapping property" as follows.

A map $V^* \times V^* \times \cdots \times V^* \xrightarrow{\psi} W$ to a vector space W is called *symmetric* if it doesn't change its value under any permutation of the arguments, and *multilinear* if it is linear in each argument when all the others are fixed. For example, the multiplication map

$$\underbrace{V^* \times V^* \times \cdots \times V^*}_{d} \xrightarrow{\mu_d} V_d \tag{1}$$

which sends a collection of linear forms $(\ell_1, \ell_2, \dots, \ell_d)$ into their product $\ell_1 \cdot \ell_2 \cdot \cdots \cdot \ell_d$ is the symmetric multilinear map. Let the map σ in the commutative diagram

$$\begin{tikzcd} & U \arrow[dd, "\varphi"] \\ V^* \times V^* \times \cdots \times V^* \arrow[ur, "\sigma"] \arrow[dr, "\psi"'] & \\ & W \end{tikzcd}$$

be symmetric and multilinear. If the vertical map φ is *linear*, then the map $\psi = \varphi\sigma$ is symmetric and multilinear. Hence, sending $\varphi \longmapsto \psi = \varphi\sigma$, we get the linear operator from the space of linear maps $U \longrightarrow W$ to the space of symmetric multilinear maps $V^* \times V^* \times \cdots \times V^* \longrightarrow W$.

[1] i. e. the coordinates in V with respect to $\{e_i\}$

[2] i. e. different polynomials always give the different polynomial functions on V

There exists the *universal multilinear map*, which is usually denoted by

$$V^* \times V^* \times \cdots \times V^* \xrightarrow{s_d} S^d(V^*) \,,$$

such that the corresponding operator $\varphi \longmapsto \psi = \varphi\sigma$ is *bijective for any vector space* W.

Exercise 1.2. Check, that the multiplication map (1) is universal.

Exercise 1.3. Show that if $V^* \times V^* \times \cdots \times V^* \xrightarrow{s} U$ and $V^* \times V^* \times \cdots \times V^* \xrightarrow{s'} U'$ are two universal symmetric multilinear maps, then there exists a unique linear isomorphism $U \xrightarrow{\vartheta} U'$ such that $s' = \vartheta\, s$.

AFFINE SPACE $\mathbb{A}(V)$ is just the set of vectors in V. The point that corresponds to the zero vector is called *an origin* and denoted by O. All other points may be pictured as the ends of non zero vectors drawn from the origin. If V is n-dimensional and the feeling of V is not essential, we will write $\mathbb{A}^n$ instead of $\mathbb{A}(V)$.

Certainly, each polynomial function on V induces the polynomial function on $\mathbb{A}(V)$. The subset $X \subset \mathbb{A}^n$ is called *closed affine algebraic subset* if it may be represented as the set of all solutions for some (possibly, infinite) system of polynomial equations $\{ f_\alpha(x) = 0 \}$.

PROJECTIVE SPACE[3] $\mathbb{P}(V)$ is the set of all 1-dimensional vector subspaces in V, or equivalently, the set of all lines passing through the origin in $\mathbb{A}(V)$. If V is $(n+1)$-dimensional, $\mathbb{P}(V)$ is called an n-dimensional projective space and is often denoted by $\mathbb{P}_n$ instead of $\mathbb{P}(V)$. To produce the usual points from the rays going from the origin we should put a screen inside $\mathbb{A}(V)$, i. e. intersect the lines forming $\mathbb{P}(V)$ with some codimension 1 affine hyperplane $U \subset \mathbb{A}(V)$ that doesn't contain the origin (like in the fig. 1.1). Such a hyperplane is called *an affine chart* in $\mathbb{P}(V)$. Of course, there is no affine chart U covering the whole projective space. The difference $U_\infty \overset{\text{def}}{=} \mathbb{P}_n \setminus U$ consists of all lines laying in the parallel copy of U passing through the origin. It is called *an infinity hyperplane* for the chart U, because it really is the projective subspace of codimension 1 in $\mathbb{P}(V)$. So, we get a decomposition of $\mathbb{P}_n$ into the disjoint union of "affine cells":

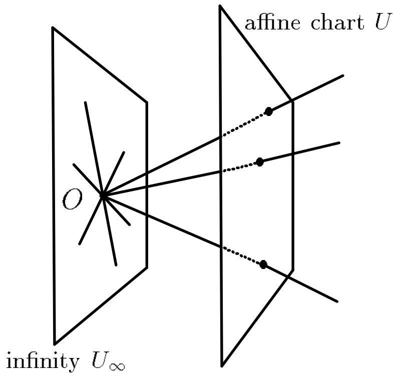

fig. 1.1. The projective world.

$$\mathbb{P}_n = U \sqcup U_\infty = \mathbb{A}^n \sqcup \mathbb{P}_{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \ldots \sqcup \mathbb{A}^0$$

Exercise 1.4*. If you have some experience in CW-topology, try to calculate integer homology groups of the real and complex projective spaces using the above cell decomposition.

For any *homogeneous* polynomial function $f \in S^d(V^*)$ the zero set $(f)_0 \overset{\text{def}}{=} \{ v \in V \mid f(v) = 0 \}$ is well defined subset in $\mathbb{P}(V)$, because of $f(v) = 0 \iff f(\lambda v) = \lambda^d f(v) = 0$. Such the subsets as well as all their intersections are called *closed projective algebraic sets*.

[3] in this section we always take $\dim V = n+1$